No part of the candidate evidence in this exemplar material may be presented in an external assessment for the purpose of gaining credits towards an NCEA qualification.

# 3

SUPERVISOR'S USE ONLY

**91577**

915770

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

# Level 3 Calculus, 2015

## 91577 Apply the algebra of complex numbers in solving problems

2.00 p.m. Wednesday 25 November 2015
Credits: Five

| Achievement | Achievement with Merit | Achievement with Excellence |
|---|---|---|
| Apply the algebra of complex numbers in solving problems. | Apply the algebra of complex numbers, using relational thinking, in solving problems. | Apply the algebra of complex numbers, using extended abstract thinking, in solving problems. |

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

**You should attempt ALL the questions in this booklet.**

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–11 in the correct order and that none of these pages is blank.

**YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.**

Not Achieved

TOTAL 6

ASSESSOR'S USE ONLY

## QUESTION ONE

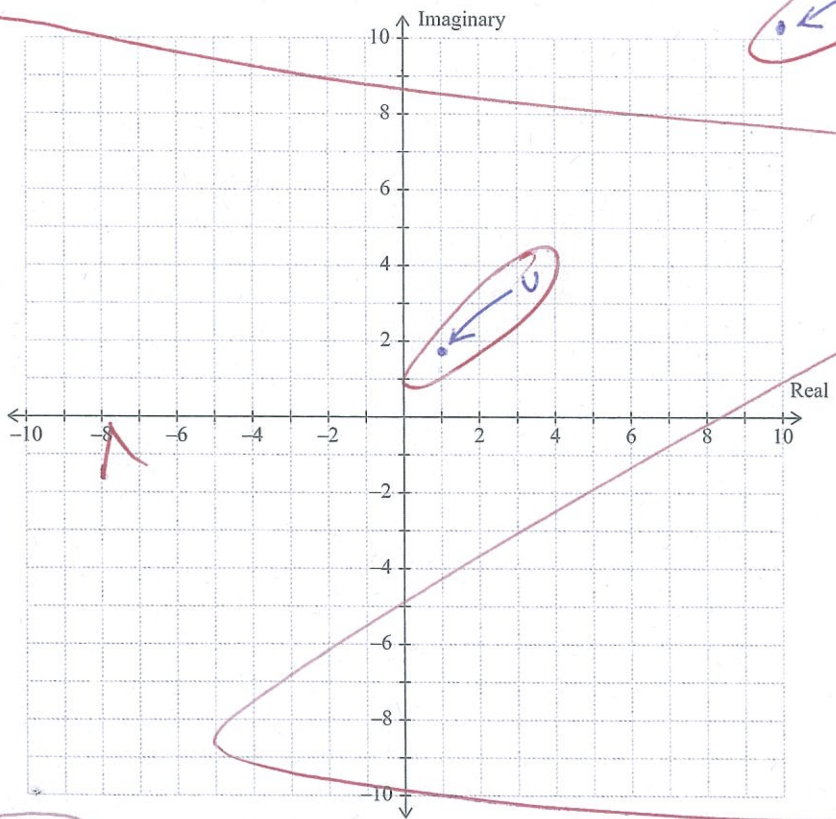
(a) Solve the equation $x^2 - 8x + 4 = 0$.

Write your answer in the form $a \pm b\sqrt{c}$, where $a$, $b$, and $c$ are integers and $b \neq 1$.

$$\underset{a}{x^2} - \underset{b}{8x} + \underset{c}{4} = 0 \qquad = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \frac{-8 \pm \sqrt{64 - 4 \times 1 \times 4}}{2} \Rightarrow \frac{-8 \pm \sqrt{48}}{2} \Rightarrow \frac{-8 \pm \sqrt{48}}{2}$$

(b) If $u = 1 + \sqrt{3}\text{i}$, clearly show $u^3$ on the Argand diagram below.




$u = 1 + \sqrt{3}i$

$u^3 = (1+\sqrt{3})(1+\sqrt{3})(1+\sqrt{3})$

$\Rightarrow (1 + \sqrt{3} + \sqrt{3} + 3)(1+\sqrt{3}) \Rightarrow (1 + 2\sqrt{3} + 3)(1 + \sqrt{3})$

$\Rightarrow 1 + \sqrt{3} + 2\sqrt{3} + (2\sqrt{3} \times \sqrt{3}) + 3 + 3\sqrt{3}$

$\Rightarrow 1 + \sqrt{3} + 2\sqrt{3} + 6 + 3 + 3\sqrt{3}$

$\Rightarrow 10 + 6\sqrt{3}i$

ASSESSOR'S USE ONLY

(c) $v$ is the complex number $3 - 7\text{i}$
$w$ is the complex number $-4 + 6\text{i}$.

Find the real numbers $p$ and $q$ such that $pv + qw = 6.5 - 11\text{i}$.

ASSESSOR'S USE ONLY

$v = 3 - 7i \quad w = -4 + 6i$

$pv + qw = 6.5 - 11i$

$\Rightarrow p(3-7i) + q(-4+6i) = 6.5 - 11i$

$\Rightarrow 3p - 7ip - 4q + 6iq - 6.5 + 11i = 0$

$\Rightarrow$

u
ns

(d) Prove that the roots of the equation $3x^2 + (2c + 1)x - (c + 3) = 0$ are always real for all values of $c$, where $c$ is real.

$3x^2 + (2c+1)x - (c+3) = 0$

$3x^2 + 2cx + x - c - 3 = 0$

$3x^2 + 2cx + x - 3 = c$

ns

(e) If $x^2 + bx + c$ and $x^2 + dx + e$ have a common factor of $(x - p)$,

prove that $\dfrac{e-c}{b-d} = p$, where $b$, $c$, $d$, $e$, and $p$ are all real.

$x^2 + bx + c$
$x^2 + dx + e$ $\rightarrow (x - p)$

ASSESSOR'S USE ONLY

N8

N2

## QUESTION TWO

(a) What is the remainder when $2x^3 + x^2 - 5x + 7$ is divided by $x + 3$?

$\cancel{2x^2 + 7x + 21}$

$\cancel{x+3 \overline{)\, 2x^3 + x^2 - 5x + 7}}$

$\cancel{2x^3 + 6x^2}$

$\cancel{7x^2 + 21x}$

$\cancel{21x + 63}$

remainder $= 23$

n

(b) The complex number $\frac{2+3i}{5+i}$ can be expressed in the form $k(1 + i)$, where $k$ is a real number.

Find the value of $k$.

$$\frac{(2+3i)}{(5+i)} \times \frac{(5-i)}{(5-i)} \Rightarrow \frac{10 - 2i + 15i - 3i^2}{25 - 5i + 5i - i^2}$$

$$\Rightarrow \frac{13 + 13i}{26} \Rightarrow \frac{13}{26} + \frac{13i}{26} \Rightarrow \frac{1}{2} + \frac{1}{2}i$$

$$K(1 + i) \Rightarrow K = \frac{1}{2} \Rightarrow \frac{1}{2}(1 + i) = \frac{1}{2} + \frac{1}{2}i$$

u

(c) Find real numbers $A$, $B$ and $C$ such that $\dfrac{1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x-1)}$

//

(d) Write the complex number $\left(\dfrac{4i^7 - i}{1+2i}\right)^2$ in the form $a + bi$, where $a$ and $b$ are real numbers.

$$\left(\frac{4i^7 - i}{1+2i}\right)^2 \Rightarrow \frac{(4i^7 - i)}{(1+2i)} \times \frac{(4i^7 - i)}{(1+2i)}$$

$$\Rightarrow \frac{16i^{14} - 4i^8 - 4i^8 + i^2}{1 + 2i + 2i + 4i^2} \Rightarrow \frac{16i^{14} - 8i^8 - 1}{-3+4i}$$

$$\Rightarrow \frac{(16i^{14} - 8i^8 - 1)}{(-3+4i)} \times \frac{(-3-4i)}{(-3-4i)} \Rightarrow \frac{-48i^{14} - 64i^{15} + 24i^8 + 32i^9 + 3 + 4i}{9 + 12i - 12i - 16i^2}$$

$$\Rightarrow \frac{-48i^{14} - 64i^{15} + 24i^8 + 32i^9 + 3 - 4i}{25}$$

//

A

ns

(e) Find the Cartesian equation of the locus described by $\arg\left(\frac{z-2}{z+5}\right)=\frac{\pi}{4}$

ASSESSOR'S USE ONLY

N2

## QUESTION THREE

ASSESSOR'S USE ONLY

(a) If $z = 4 + 2i$ and $w = -1 + 3i$, find $\arg(zw)$.

$z = 4 + 2i \quad w = -1 + 3i$

$zw = (4+2i)(-1+3i) = -4 + 12i - 2i + 6i^2$

$= -10 + 10i$

A

NS

(b) For what real value(s) of $k$ does the equation $kx^2 + \frac{x}{k} + 2 = 0$ have equal roots?

$Kx^2 + \frac{x}{K} + 2 = 0$

$K^2x^2 + x + 2K = 0$

A

NS

(c) One solution of the equation $3w^3 + Aw^2 - 3w + 10 = 0$ is $w = -2$.

If $A$ is a real number, find the value of $A$ and the other two solutions of the equation.

$3w^3 + Aw^2 - 3w + 10 = 0$

One solution $= W = -2$

$\Rightarrow 3(-2)^3 + A(-2)^2 - 3(-2) + 10 = 0$

$\Rightarrow -24 + 4A + 6 + 10 = 0$

$\Rightarrow 4A - 8 = 0$

$\Rightarrow 4A = 8 \qquad A = 2$

$3w^3 + 2w^2 - 3w + 10 = 0$

Other solutions : $x =$

u

ns

(d) Solve the equation $z^3 = k + \sqrt{3}\,k\mathrm{i}$, where $k$ is real and positive.

Write your solutions in polar form in terms of $k$.

ASSESSOR'S USE ONLY

$z^3 = K + \sqrt{3}Ki$

$z = x + iy$

$\Rightarrow (x+iy)(x+iy)(x+iy) \Rightarrow (x^2 + xiy + xiy + i^2y^2)(x+iy)$

$\Rightarrow x^3 + x^2iy + x^2iy + xi^2y^2 + x^2iy + xi^2y^2 + xi^2y^2 + i^3y^3$

$\Rightarrow x^3 + i^3y^3 + 3x^2iy + 3xi^2y^2 = K + \sqrt{3}Ki$

n

**Question Three continues on the following page.**

(e) (i) Find each of the roots of the equation $z^5 - 1 = 0$.

ASSESSOR'S USE ONLY

$z^5 - 1 = 0$

N8

(ii) Let $p$ be the root in part (i) with the smallest positive argument.

Show that the roots in part (i) can be written as $1, p, p^2, p^3, p^4$.

//

N2

| Not Achieved exemplar for 91577 2015 | | | Total score | 06 |
|---|---|---|---|---|
| **Q** | **Grade score** | **Annotation** | | |
| 1 | N2 | This question provides evidence for N2 because the candidate has made a successful start to solving the problem in part c.<br><br>a) The solution for the quadratic equation is incorrect. This candidate needed to be more accurate in their substitution into the quadratic formula since –b should have been positive 8 and continued on to provide an answer in its most simplified surd form.<br><br>b) The i of $1 + \sqrt{3}i$ has been lost from the first line of the three bracket expansion so the complex number found for $u^3$ is incorrect. The placement of $u^3$ on the Argand diagram is consistent with their working.<br><br>c) The candidate has successfully substituted for the complex numbers, v and w, in the given equation as well as accurately expanding the brackets. To make progress in this part of question one, the student needed to realise that they should equate the real and imaginary parts.<br><br>d) The candidate does not realise that they need to be investigating the discriminant.<br><br>e) The candidate does not realise that the given common factor means that they could be using the Factor Theorem to write two equations involving p equal to zero. | | |
| 2 | N2 | This question provides evidence for N2 because the candidate has successfully completed part b).<br><br>a) The remainder provided is incorrect. It should have been -23.<br><br>b) The candidate has successfully rewritten the given quotient of complex numbers, $\frac{2+3i}{5+i}$ , with a rational denominator and simplified their answer to: $\frac{1}{2}(1+i)$, which allowed them to be able to identify the value of the unknown, k.<br><br>c) Not attempted.<br><br>d) The working in 2d is on track. If the beginning of the second to last line had been simplified to $\frac{-25}{-3+4i}$, this candidate would have achieved another u grade for this question. If they had transferred the +4i at the end of the second to last line down as it was rather than -4i as they have written, their numerator would have simplified to the correct 75+100i and they could have arrived at the final correct answer of 3+4i. They were on the right track for a M5 for this question.<br><br>e) Not attempted. | | |
| 3 | N2 | This question is an N2 because the candidate successfully found the value of A in c).<br><br>a) The candidate has found the complex number which represented *zw*. As was common with other candidates, they were not able to continue to work out the argument of the complex number, $-10 + 10i$.<br><br>b) The candidate did not recognise that they needed to find the discriminant to make progress with this problem.<br><br>c) The candidate has correctly applied the factor theorem to the given cubic equation to find the value of the unknown pronumeral, A. They have not gone on to find the resulting quadratic factor or its roots.<br><br>d) The student has not recognised the need to convert the given complex number to polar form so that they could use de Moivre's Theorem to find the required roots.<br><br>e) Not attempted. | | |