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SUPERVISOR'S USE ONLY

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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2015

91577 Apply the algebra of complex numbers in solving problems

2.00 p.m. Wednesday 25 November 2015
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–11 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Merit

TOTAL 16

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QUESTION ONE

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(a) Solve the equation $x^2 - 8x + 4 = 0$.

Write your answer in the form $a \pm b\sqrt{c}$, where a, b, and c are integers and $b \neq 1$.

$x^2 - 8x = -4$

$\cancel{x^2}\ x^2 - 8x + 16 = -4 + 16$

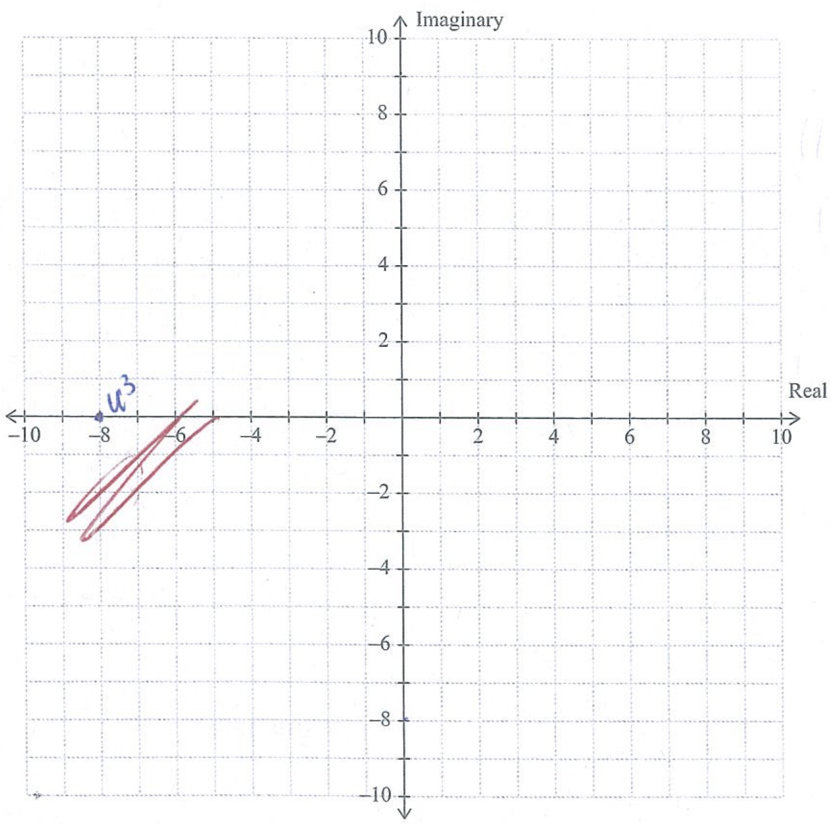
$(x-4)^2 = 12$

$x - 4 = \pm 2\sqrt{3}$

$x = 2\sqrt{3} + 4$ or $-2\sqrt{3} + 4$

u

(b) If $u = 1 + \sqrt{3}\text{i}$, clearly show u^3 on the Argand diagram below.



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$(1+\sqrt{3}i)^2(1+\sqrt{3}i)$

$[1 + 2\sqrt{3}i + (\sqrt{3}i)^2](1+\sqrt{3}i)$

$\cancel{1 + 2\sqrt{3}i - 3}$

$(2\sqrt{3}i - 2)(1+\sqrt{3}i)$

$2\sqrt{3}i\ \cancel{+}-6 - 2 - 2\sqrt{3}i$

$u^3 = -8$

(c) v is the complex number $3 - 7i$
w is the complex number $-4 + 6i$.

Find the real numbers p and q such that $pv + qw = 6.5 - 11i$.

$v+w = 3-7i-(4+6i)$
$= 3-7i-4-6i$
$= -1-13i$

$p(3-7i)+q(-4+6i) = 6.5-11i$
$3p-7pi \cancel{+} -4q+6qi = 6.5-11i$
$3p-4q \quad -7pi+6qi = 6.5-11i$

$\begin{cases} 3p-4q = 6.5 \\ -7pi+6qi = -11i \end{cases}$

$\therefore \begin{cases} p = 0.5 \\ q = -1.25 \end{cases}$

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(d) Prove that the roots of the equation $3x^2 + (2c + 1)x - (c + 3) = 0$ are always real for all values of c, where c is real.

$\Delta = b^2 - 4ac$
$= (2c+1)^2 + 12(c+3)$
$= 4c^2+4c+1+12c+36$
$= 4c^2+16c+37$
$= 4(c^2+4c)+37$
$= 4(c^2+4c+4-4)+37$
$= 4(c^2+4c+4)-16+37$
$= 4(c+2)^2+21$

$\because (c+2)^2 > 0$
$\therefore \Delta = 4(c+2)^2+21 > 0$
$\therefore$ The roots are always real

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(e) If $x^2 + bx + c$ and $x^2 + dx + e$ have a common factor of $(x - p)$,

prove that $\frac{e-c}{b-d} = p$, where b, c, d, e, and p are all real.

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$p^2 + bp + c = p^2 + dp + e$

$pb - pd = e - c$

$p(b-d) = e - c$

$\frac{e-c}{b-d} = p$

$e-c = pb - pd$

Pencil

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M6

QUESTION TWO

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(a) What is the remainder when $2x^3 + x^2 - 5x + 7$ is divided by $x + 3$?

$$x+3 \overline{\smash{)}\, 2x^3+x^2-5x+7} \quad 2x^2-5x+10$$
$$-(2x^3+6x^2)$$
$$-5x^2-5x$$
$$-(-5x^2-15x)$$
$$10x+7$$
$$-(10x+30)$$
$$-21$$

$\therefore$ remainder is -21.

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(b) The complex number $\frac{2+3i}{5+i}$ can be expressed in the form $k(1 + i)$, where k is a real number.

Find the value of k.

$$= \frac{(2+3i)(5-i)}{(5+i)(5-i)}$$
$$= \frac{10-2i+15i-3i^2}{26}$$
$$= \frac{13+13i}{26}$$
$$= \frac{1}{2}+\frac{1}{2}i$$
$$= \frac{1}{2}(1+i)$$
$$\therefore k = \frac{1}{2}$$

u

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(c) Find real numbers A, B and C such that $\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)}$

$$\frac{Ax(x-1)}{x^2(x-1)} + \frac{B(x-1)}{x^2(x-1)} + \frac{Cx^2}{x^2(x-1)}$$

$$\frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)} = \frac{1}{x^2(x-1)}$$

$$Cx^2 + Ax(x-1) + B(x-1) = 1$$

$$Cx^2 + Ax^2 - A\,x + Bx - B = 1$$

$$(A+C)x^2 + Bx - (1+A+B) = 0$$

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(d) Write the complex number $\left(\frac{4i^7 - i}{1+2i}\right)^2$ in the form $a + bi$, where a and b are real numbers.

$$\left(\frac{4i^6 \cdot i - i}{1+2i}\right)^2$$

$$\left(\frac{-4i - i}{1+2i}\right)^2$$

$$\left(\frac{-5i}{1+2i}\right)^2$$

$$\frac{25i^2}{(1+2i)^2}$$

$$\frac{-25}{1+4i+4i^2}$$

$$\frac{-25}{-3+4i}$$

$$\frac{-25(-3-4i)}{9+4}$$

$$\frac{75+100i}{13} = \frac{75}{13} + \frac{100}{13}i$$

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ASSESSOR'S USE ONLY

(e) Find the Cartesian equation of the locus described by $\arg\left(\frac{z-2}{z+5}\right)=\frac{\pi}{4}$

let $z = x+yi$

$\arg\left(\frac{x+yi-2}{x+yi+5}\right)=\frac{\pi}{4}$

$\arg\left(\frac{x-2+yi}{x+5+yi}\right) = \cancel{\tan}\ \frac{\pi}{4}$.

$\left|\frac{x-2+yi}{x+5+yi}\right| = \tan\frac{\pi}{4} = 1$

$\frac{\sqrt{(x-2)^2+y^2}}{\sqrt{(x+5)^2+y^2}} = 1$

$(x-2)^2+y^2 = (x+5)^2+y^2$

$x^2 \cancel{-4x+4+y^2}$

$\therefore (2,0), (-5,0)$

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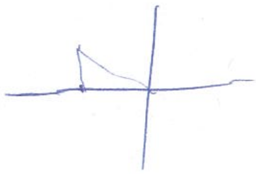
A4

QUESTION THREE

ASSESSOR'S USE ONLY

(a) If $z = 4 + 2i$ and $w = -1 + 3i$, find $\arg(zw)$.



$zw = (4+2i)(-1+3i)$

$= -4 + 12i - 2i - 6$

$= -10 + 10i$

$\arg(zw) = \tan^{-1}\left(\frac{10}{-10}\right) = \phi$

$\arg(zw) = 45^\circ + 90^\circ = 135^\circ$

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(b) For what real value(s) of k does the equation $kx^2 + \frac{x}{k} + 2 = 0$ have equal roots?

~~$x_1 = x_2$~~

~~$\frac{-\frac{1}{k} + \sqrt{\frac{1}{k^2} - 8k}}{2k} = \frac{-\frac{1}{k} - \sqrt{\frac{1}{k^2} - 8k}}{2k}$~~

$\sqrt{\frac{1}{k^2} - 8k} =$

$\Delta = b^2 - 4ac = 0$

$\frac{1}{k^2} - 8k = 0$

$\frac{1}{k^2} = 8k \Rightarrow k^3 = \frac{1}{8}$

$\frac{1}{k^3} = 8 \qquad k = \frac{1}{2}$

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(c) One solution of the equation $3w^3 + Aw^2 - 3w + 10 = 0$ is $w = -2$.

If A is a real number, find the value of A and the other two solutions of the equation.

$3 \times (-2)^3 + 4A + 6 + 10 = 0$

$-24 + 4A + 6 + 10 = 0$

$A = 2$

$$\begin{array}{r} 3w^2 - 4w + 5 \\ w+2 \;\big)\; 3w^3 + 2w^2 - 3w + 10 = 0 \\ -(3w^3 + 6w^2) \\ \hline -4w^2 - 3w \\ -(-4w^2 - 8w) \\ \hline 5w + 10 \\ 5w + 10 \\ \hline 0 \end{array}$$

$3w^2 - 4w + 5 = 0$

$w^2 - \frac{4}{3}w + \frac{5}{3} = 0$

$w^2 - \frac{4}{3}w + \frac{4}{9} - \frac{4}{9} + \frac{5}{3}$

$(w - \frac{2}{3})^2 = -\frac{11}{9}$

$(w - \frac{2}{3}) = \sqrt{\frac{11}{9} i^2}$

$\begin{cases} w_1 = \frac{2}{3} + \frac{\sqrt{11}}{3} i \\ w_2 = \frac{2}{3} - \frac{\sqrt{11}}{3} i \end{cases}$

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(d) Solve the equation $z^3 = k + \sqrt{3}\,k\mathrm{i}$, where k is real and positive.

Write your solutions in polar form in terms of k.

$z^3 = 2(\cos 60° + i\sin 60°)$

$z = 2^{\frac{1}{3}}(\cos 60 \times \tfrac{1}{3} + i\sin 60 \times \tfrac{1}{3})$

$z_1 = \sqrt[3]{2}(\cos 20° + i\sin 20°)$

$z_2 = \sqrt[3]{2}(\cos 140° + i\sin 140°)$

$z_3 = \sqrt[3]{2}(\cos 260° + i\sin 260°)$

ASSESSOR'S USE ONLY

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Question Three continues on the following page.

(e) (i) Find each of the roots of the equation $z^5 - 1 = 0$.

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$z^5 = 1 + 0i$

$z^5 = \cos 0° + i\sin 0°$

$z_1 = \cos 0° + i\sin 0°$

$z_2 = \cos 72° + i\sin 72°$

$z_3 = \cos 144° + i\sin 144°$

$z_4 = \cos 216° + i\sin 216°$

$z_5 = \cos 288° + i\sin 288°$

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(ii) Let p be the root in part (i) with the smallest positive argument.

Show that the roots in part (i) can be written as $1, p, p^2, p^3, p^4$.

$p = \cos 0° + i\sin 0° = 1$

$p = \cos 72° + i\sin 702°$

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M6

Merit exemplar for 91577 2015		Total score	16
Q	**Grade score**	**Annotation**	
1	M6	This question provides evidence for an M6 because the candidate has correctly completed the two merit level questions c) and d). a) The candidate has solved the quadratic equation by completing the square and then written their solution in its most simplified form. b) u^3 has been found by expanding the three brackets and then clearly shown on the Argand diagram. c) The correct pair of equations in two unknowns has been found by substituting for v and w into the given equation and equating the real and imaginary parts of the two complex numbers. The candidate has then solved the two equations simultaneously to find the values of p and q. d) This candidate has not only realised that the discriminant must be positive if the quadratic equation is always to have real roots but also that they can prove this by writing the discriminant in perfect square form. e) The candidate has not made any effort to explain why they have written their first line of working. To gain a t in this question, the candidate had to clearly communicate the concepts required to solve the problem. When using this approach, the candidate needed to explain how the factor theorem would have allowed them to arrive at this point. The rest of the working is correct, hence the r grade.	
2	A4	This question provides evidence for A4 because the candidate has gained three u grades in parts b), d) and e). a) The working for the long division is mainly correct, however the last calculation: 7-30 has resulted in -21 which is incorrect. b) The quotient of the two complex numbers written in rectangular form was simplified correctly to arrive at a rational denominator and the k value was identified. c) The candidate has attempted to write the three fractions over a common denominator. However there is an error in their working and they have shown that they do not know how to finish the problem. d) The first six lines of the working are correct. However the final denominator is incorrect when they tried to multiply -3+4i with its conjugate. The result should have been 25 not 13. e) Little progress has been made on this question because the candidate has confused arguments with moduli.	
3	M6	This question provides evidence for M6 because the candidate has gained two r grades in parts c) and ei). a) The candidate has not only found the product, zw, they have also shown a good understanding of how to find an argument. b) The discriminant has been found and solved equal to zero. c) The factor theorem has been applied to find the value of A = 2 and the other two roots have also been found using algebraic long division and the method of completing the square. d) This candidate has come close to finding the 3 cube roots of the equation. The arguments of the three solutions are correct, hence the u grade but the candidate was not able to find the modulus of the given complex number so the moduli of the cube roots are also incorrect. e) The candidate has found the 5 roots of the quintic equation by using De Moivre's Theorem. They have not been able to identify p, the root with the smallest positive argument, or show that three of the other roots can be expressed as powers of p.	