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Level 3 Calculus, 2016

91577 Apply the algebra of complex numbers in solving problems

9.30 a.m. Wednesday 23 November 2016
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–8 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 24

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QUESTION ONE

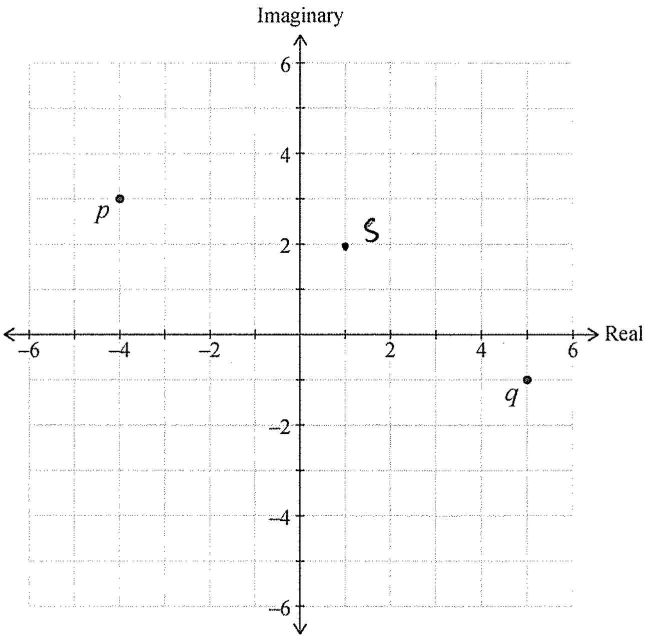
(a) Complex numbers p and q are represented on the Argand diagram.

If $s = p + q$, then show s on the Argand diagram below.

$P = -4+3i$

$q = 5 - i$

$\therefore S = 1+2i$



(b) Dividing $2x^3 + 5x^2 + Ax + 7$ by $x + 3$ gives a remainder of 16.

What is the value of A?

let $P(x) = 2x^3 + 5x^2 + Ax + 7$

$\therefore P(-3) = 16$

$\therefore 2(-3)^3 + 5(-3)^2 + A(-3) + 7 = 16$

$\therefore A = -6$

(c) Solve the equation $5 - \sqrt{x} = \sqrt{x - p}$ for x in terms of p.

$5 - \sqrt{x} = \sqrt{x-p}$ $\quad$ $; x > p \quad , x > 0.$

$25 - 10\sqrt{x} + x = x - p$

$25 + p = 10\sqrt{x}$

$\therefore \sqrt{x} = \frac{(p+25)}{10}$

$\therefore x = \frac{(p+25)^2}{100}$

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(d) If $w = 1 + 2i$, find the value of $w^2 + \frac{w}{\bar{w}}$, giving your answer in the form $a + bi$, where a and b are real.

You must clearly show each step of your working.

• $w = 1+2i \quad \therefore \quad w^2 = (1+2i)^2 = -3+4i$

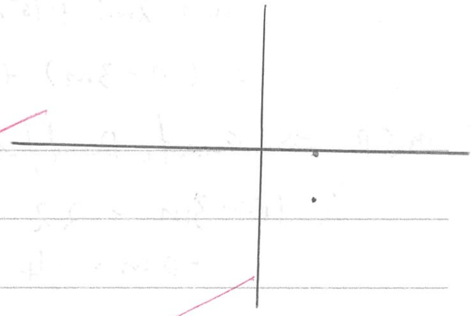
• $\bar{w} = 1-2i \quad \therefore \quad \frac{w}{\bar{w}} = \frac{1+2i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{-3+4i}{5} = -\frac{3}{5} + \frac{4}{5}i$

$\therefore \quad w^2 + \frac{w}{\bar{w}} = -3+4i - \frac{3}{5} + \frac{4}{5}i$

$= \underline{\frac{-18}{5} + \frac{24}{5}i}$ or $-3.6 + 4.8i$ //

(e) The locus described by $|z - 2 + 3i| = |z - 1|$ is a straight line.

Find the gradient of that line.



let $z = x+iy$

$\therefore |x+iy-2+3i| = |x+iy-1|$

$\Leftrightarrow |(x-2)+i(y+3)| = |(x-1)+i(y)|$

$\Leftrightarrow \sqrt{(x-2)^2+(y+3)^2} = \sqrt{(x-1)^2+y^2}$

$\Rightarrow (x-2)^2+(y+3)^2 = (x-1)^2+y^2$

$\Leftrightarrow \cancel{x^2}-4x+4+\cancel{y^2}+6y+9 = \cancel{x^2}-2x+1+\cancel{y^2}$

$\Leftrightarrow -4x+6y+13 = -2x+1$

$\Leftrightarrow 6y = 2x-12$

$\Leftrightarrow y = \frac{1}{3}x - 2$.

therefore the corresponding perpendicular bisector joining points $(2,-3)$ and $(1,0)$ is a locus of $y = \frac{1}{3}x - 2$ and has a gradient of $\frac{1}{3}$. //

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QUESTION TWO

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(a) Solve the equation $x^2 - 6x + 12 = 0$.

Write your answer in the form $a \pm \sqrt{b}\,\mathrm{i}$, where a and b are rational numbers.

$x^2 - 6x + 12 = 0$

$\Leftrightarrow x^2 - 6x + 9 = -12 + 9$

$\Leftrightarrow (x-3)^2 = -3$

$\Leftrightarrow x = 3 \pm \sqrt{3}\,i$

(b) $u = 2 + 3\mathrm{i}$ and $v = 5 + m\mathrm{i}$.

Find the value of m if $uv = 22 + 7\mathrm{i}$.

$uv = (2+3i)(5+mi)$

$= 10 + 2mi + 15i - 3m$

$= (10 - 3m) + i(2m + 15) = 22 + 7i$

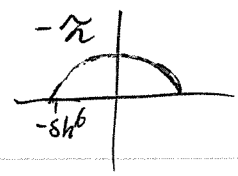
$m \in \mathbb{R} \Rightarrow$ Equate Real / Imaginary

ie $10 - 3m = 22$ or $2m + 15 = 7$

$\Rightarrow m = -4$ $\therefore m = -4$ so confirms.

(c) Solve the equation $z^3 = -8k^6$, where k is real.

Write your solutions in polar form in terms of k.



$\therefore z^3 = 8k^6 \operatorname{cis}(\pi)$

$z = 2k^2 \operatorname{cis}\left(\frac{\pi}{3} + \frac{2\pi n}{3}\right)$ $n \in \mathbb{Z}$.

$\therefore z_1 = 2k^2 \operatorname{cis}\left(\frac{\pi}{3}\right)$ $n = 0$

$z_2 = 2k^2 \operatorname{cis}(\pi)$ $n = 1$

$z_3 = 2k^3 \operatorname{cis}\left(-\frac{\pi}{3}\right)$ $n = -1$

(d) Prove that $\left|\frac{4+2i}{1+i}\right| = \sqrt{10}$.

You must clearly show each step of your working.

• Consider $t = \frac{4+2i}{1+i} \times \frac{1-i}{1-i} = \frac{6-2i}{2} = 3-i$

$\therefore |t| = \sqrt{(3)^2+(-1)^2}$

$= \sqrt{9+1}$

$= \sqrt{10}$ QED.

(e) Find the value of k if the equation $8-x+2\sqrt{2x+k}=0$ has equal roots.

$8-x+2\sqrt{2x+k} = 0$

$2\sqrt{2x+k} = x-8$

$4(2x+k) = (x-8)^2$

$8x+4k = x^2-16x+64$

$x^2-24x+(64-4k) = 0$

for equal roots $\Delta = 0 \therefore b^2-4ac=0$

$\therefore (-24)^2-4(64-4k) = 0$

$576-256+16k = 0$

$\therefore 16k = -320$

$\therefore k = -20$

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QUESTION THREE

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(a) Write $\frac{5}{2+\sqrt{3}}$ in the form $a+b\sqrt{c}$.

$\frac{5}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{10-5\sqrt{3}}{1}$

$= 10-5\sqrt{3}$

(b) If $v = 4\operatorname{cis}\frac{3\pi}{4}$ and $w = 6\operatorname{cis}\frac{2\pi}{3}$, write the exact value $\frac{v}{w}$ in polar form.

$\therefore \frac{v}{w} = \frac{4}{6} \operatorname{cis}\left(\frac{3\pi}{4} - \frac{2\pi}{3}\right)$

$= \frac{2}{3} \operatorname{cis}\left(\frac{\pi}{12}\right)$

(c) $z = 3 - 4\mathrm{i}$ is one solution of the equation

$z^3 - 8z^2 + \mathrm{B}z - 50 = 0.$

Find the value of B.

if $z_1 = 3-4i$ then $z_2 = 3+4i$ by conjugate root theorem $\therefore$ by vietas formulas;

$\alpha+\beta+\gamma = z_1 + z_2 + z_3 = \frac{-b}{a}$

$\therefore 3-4i + 3+4i + z_3 = 8$

$\therefore z_3 = 2$

$\therefore \alpha\beta + \alpha\gamma + \beta\gamma = z_1z_2 + z_1z_3 + z_2z_3 = \frac{B}{1}$

$(3-4i)(3+4i) + (3-4i)(2) + (3+4i)(2) = B$

$\therefore B = 37.$

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(d) If u and v are complex numbers, prove that $\overline{uv} = \bar{u} \cdot \bar{v}$.

let $u = a+bi$, $v = c+di$.

$\therefore \quad uv = (a+bi)(c+di)$

$= ac + adi + bci - bd$

$= ac - bd + i(ad+bc)$

$\therefore \quad \overline{uv} = ac - bd - i(ad+bc)$

$\bar{u} \cdot \bar{v} = (a-bi)(c-di)$

$= ac - adi - bci - bd$

$= ac - bd - i(ad+bc)$

$= \overline{uv}$

$\therefore$ QED since LHS = RHS.

(e) u and v are two complex numbers, such that $|u+v|^2 = |u-v|^2$.

Prove that $u\bar{v}$ is purely imaginary.

let $u = a+bi$, $v = c+di$.

$\therefore |u+v|^2 = |u-v|^2$

$|a+bi+c+di| = |a+bi-c-di|$

$|(a+c)+i(b+d)| = |(a-c)+i(b-d)|$

$\left(\sqrt{(a+c)^2 + (b+d)^2}\right)^2 = \left(\sqrt{(a-c)^2 + (b-d)^2}\right)^2$

$\therefore \cancel{a^2} + 2ac + \cancel{c^2} + \cancel{b^2} + 2bd + \cancel{d^2} = \cancel{a^2} - 2ac + \cancel{c^2} + \cancel{b^2} - 2bd + \cancel{d^2}$

$\therefore \quad 4ac + 4bd = 0$

$\therefore \quad ac + bd = 0 \quad \Rightarrow$ condition.

$u\bar{v} = (a+bi)(c-di)$

$= ac - adi + bci - bdi^2$

$= ac + bd + i(bc - ad)$

but $ac + bd = 0 \quad \therefore \quad u\bar{v} = i(bc-ad)$

which is purely imaginary $\Rightarrow$ argument of $\left[\frac{\pi}{2}\right]$

QED.

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Extra paper if required.
Write the question number(s) if applicable.

QUESTION NUMBER

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Annotated Exemplar Template

Excellence exemplar 2016

Subject:	Calculus	Standard:	91577	Total score:	24

Q	Grade score	Annotation
1	E8	This question provides evidence for E8 because the candidate has gained 1 e grade for their efforts in part e) a) The position of s on the Argand diagram is clearly identified. b) The candidate has correctly found A by substituting f(-3) = 16. c) The candidate has correctly rearranged to give x in terms of p d) The candidate has the correct solution and there is sufficient evidence of algebraic manipulation. e) The candidate has correctly use the general expressions for the moduli, gone on to calculate the equation of the locus and explicitly stated the gradient of the line.
2	E8	This question provides evidence for E8 because the candidate has gained 1 e grade for their efforts in part e) a) The candidate has the correct expression for $a \pm \sqrt{b}i$. b) The candidate has the correct value for m. c) The candidate has correctly used De Moivre's Theorem to identify all 3 roots. d) The complex expression has been simplified and the modulus has been correctly calculated. e) The equation has been rearranged into a quadratic, and k has been found by solving for the discriminant equal to 0.
3	E8	This question provides evidence for E8 because the candidate has gained 1 e grade for their efforts in part e) a) The denominator has been successfully rationalised and the solution given in the correct form. b) The complex numbers have been successfully divided in polar form, and the answer is exact. c) The missing factor has been found using a variant of sums and products of roots and thence B. d) A clear, easy to follow proof, keeping generalised complex numbers. e) Moduli found and condition found. Shows that the conjugate of $u\bar{v}$ is purely imaginary although the last statement is only true if bc > ad.