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Level 3 Calculus, 2016

91577 Apply the algebra of complex numbers in solving problems

9.30 a.m. Wednesday 23 November 2016
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–8 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Merit

TOTAL 15

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QUESTION ONE

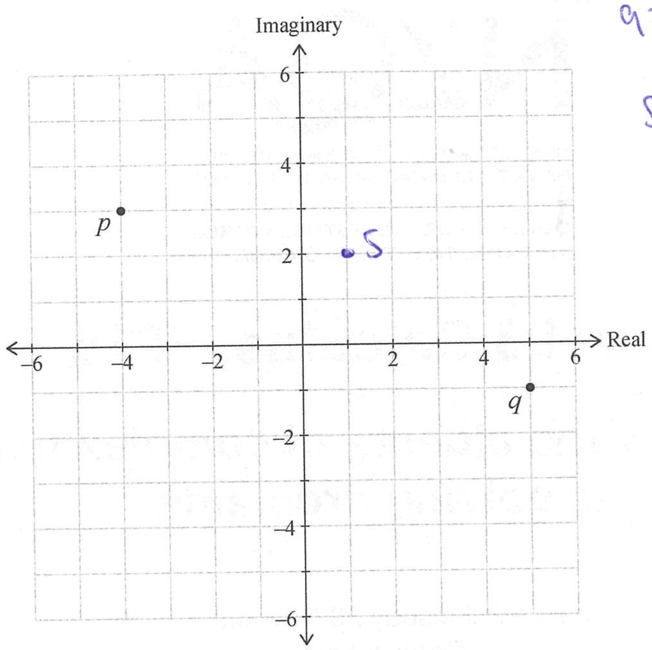
(a) Complex numbers p and q are represented on the Argand diagram.

If $s = p + q$, then show s on the Argand diagram below.

$p = -4 + 3i$

$q = 5 - i$

$s = 1 + 2i$



(b) Dividing $2x^3 + 5x^2 + Ax + 7$ by $x + 3$ gives a remainder of 16.

What is the value of A?

$16 = 2(-3)^3 + 5(-3)^2 + -3A + 7$

$16 = -54 + 45 + -3A + 7$ $\quad 16 = -3A - 2$

$18 = -3A. \quad A = -6.$

(c) Solve the equation $5 - \sqrt{x} = \sqrt{x - p}$ for x in terms of p.

$(5 - \sqrt{x})^2 = x - p. \quad 25 - 10\sqrt{x} + x = x - p.$

$25 + \cancel{5AB} \cdot 10\sqrt{x} = \cancel{-}p.$

$10\sqrt{x} = p - 25 \quad x = \left(\frac{p - 25}{10}\right)^2$

$x = \frac{p^2 - 50p + 125}{100}$

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(d) If $w = 1 + 2\text{i}$, find the value of $w^2 + \frac{w}{\overline{w}}$, giving your answer in the form $a + b\text{i}$, where a and b are real.

You must clearly show each step of your working.

$(1+2i)^2 + \frac{1+2i}{1-2i}$

$-3+4i + \frac{1+2i}{1-2i} \times \frac{1+2i}{1+2i} =$

$-3+4i + \frac{-3+4i}{5}$.

$-3.6 + 4.8i$

r

(e) The locus described by $|z - 2 + 3\text{i}| = |z - 1|$ is a straight line.

Find the gradient of that line.

$z = x + iy$ $\quad |x+iy-2+3i| = |x+iy-1|$

$= |(x-2)+(y+3)i| + |(x-1)+iy|$

$= \sqrt{(x-2)^2 + (y+3)^2} + \sqrt{(x-1)^2 + y^2}$

$\sqrt{x^2-4x+4 + y^2+6y+9} + \sqrt{x^2 2x+1 + y^2} = 0$.

$x^2 - 4x+4 + y^2 + 6y + 9 + x^2 - 2x + 1 + y^2 = 0$.

$6y + -6x + 14$

$6y = -6x + 14$

$y = -x + 14/6$. $\quad y = -x + 7/3$.

gradient $= -1$

u

M5

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QUESTION TWO

(a) Solve the equation $x^2 - 6x + 12 = 0$.

Write your answer in the form $a \pm \sqrt{b}\,\mathrm{i}$, where a and b are rational numbers.

$(x-3)^2 = -\cancel{10}12 + 9$

$(x-3) = \sqrt{-3}$ $\quad x = \sqrt{-3} \mp 3$

$x = \sqrt{3}i \pm 3$ $\quad x = 3 \pm \sqrt{3}\,i$

(b) $u = 2 + 3\mathrm{i}$ and $v = 5 + m\mathrm{i}$.

Find the value of m if $uv = 22 + 7\mathrm{i}$.

$10 \quad (2+3i)(5+mi)$

$2mi \quad 15i \quad -3m.$

$(2+3i)(5+mi) = 22+7i$ $\quad \cancel{10 + 3mi^2 + 2mi + 15i}$

$10 - 3m + 2mi + 15i = 22+7i$

$-3m + 2mi = 12 - 8i$ $\quad -3m - 12 = 2mi - 8i$

$3m + 12 = 2mi - 8i$ $\quad 3(m+4) = 2i(m-8)$

(c) Solve the equation $z^3 = -8k^6$, where k is real.

Write your solutions in polar form in terms of k.

$\cancel{z = \frac{-8}{3}k^3 \quad z^3 = -8k^6 + 0i \quad z = \frac{-8}{3}k^3}$

$\cancel{r(\cos\theta + i\sin\theta) \quad r = \sqrt{(8k^6)^2} \quad r = 8k^6}$

$\cancel{\theta = \tan^{-1}(}$

$z = -2k^3$ $\quad \cancel{8k} \; -8k^6(\cos 0 + i\sin 0)$

$r = \sqrt{(2k^3)^2 + 0i^2}$ $\quad r = 2k^3$ $\quad \tan^{-1}\frac{-8k^6}{0}$ $\quad \theta = 0$

$2k^3(\cos\theta + i\sin\theta)$ $\quad \theta = 0$

$(2k^3 \operatorname{cis} 0)^3$ $\quad 8k^6 \operatorname{cis}$

(d) Prove that $\left|\frac{4+2i}{1+i}\right| = \sqrt{10}$.

You must clearly show each step of your working.

LHS. $\frac{4+2i}{1+i} \times \frac{1-i}{1-i}$ $\frac{6-2i}{2}$ = $|3-i|$ =

LHS $\sqrt{3^2+(-1)^2} = \sqrt{10}$ $\sqrt{10} = \sqrt{10}.$

me

LHS = RHS.

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r

(e) Find the value of k if the equation $8 - x + 2\sqrt{2x+k} = 0$ has equal roots.

$\frac{8-x}{2} = -\sqrt{2x+k}$

$\left(4 - \frac{x}{2}\right)^2 = 2x + k.$ $16 - \frac{x^2}{4} = -(2x+k)$

$16 - \frac{x^2}{4} + 2x + k = 0.$

$64 - x^2 + 8x + 4k = 0.$ $x^2 - 8x - 64 - 4k = 0.$

$b^2 - 4ac = 0.$ $c = -64 - 4k.$

$b = 8$ $a = 1$

$64 - [4 \times 1 \times (-64 - 4k)]$ ~~$4k - 16k = -192$~~

~~$k = 12$~~

$-192 + (4k \times 4) = 0.$ $-192 + 16k = 0.$

$k = 12.$

M5

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QUESTION THREE

(a) Write $\frac{5}{2+\sqrt{3}}$ in the form $a+b\sqrt{c}$.

$\frac{5}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{10-5\sqrt{3}}{4-9} \quad \frac{10-5\sqrt{3}}{-5}$

$= -2 + \sqrt{3}$

(b) If $v = 4\,\text{cis}\frac{3\pi}{4}$ and $w = 6\,\text{cis}\frac{2\pi}{3}$, write the exact value $\frac{v}{w}$ in polar form.

$4 - 8$

$\frac{4\,\text{cis}\frac{3\pi}{4}}{6\,\text{cis}\frac{2\pi}{3}} = \frac{2}{3}\,\text{cis}\left(\frac{3\pi}{4} - \frac{2\pi}{3}\right)$

$= \frac{2}{3}\,\text{cis}\left(\frac{\pi}{12}\right)$

(c) $z = 3 - 4i$ is one solution of the equation

$z^3 - 8z^2 + Bz - 50 = 0.$

Find the value of B.

$(3-4i)^3 - 8(3-4i) + B(3-4i) \cancel{-50=0}$

$\cancel{-117-44i - 24+32i + (3-4i)B - 50 = 0}$

$\cancel{-141 \quad -12i = (3-4i)B}$

$z = (3-4i)$ one solution. other $= (3+4i)$

$[z-(3-4i)] \times [z-(3+4i)] = z^2 - (3-4i)z - (3+4i)z + (3+4i)(3-4i)$

$= (z^2 - 6z + 25) \; (z-2) \quad \frac{50}{25} = 2$

$\rightarrow z-2$ is other factor

$= z^3 - 6z^2 + 12z + 25z - 2z^2 - 50$

$z^3 - 2z^2 - 6z^2 + 12z + 25z - 50 = 0 \quad z^3 - 8z^2 + 37z - 50$

$B = 37$

(d) If u and v are complex numbers, prove that $\overline{uv} = \bar{u} \cdot \bar{v}$.

$\bar{u} \times \bar{v}$ Say $u = 1+2i$ $v = 2+2i$

$\boxed{1+2i \times 2+2i = -2+6i = uv}$ $\overline{uv} = -2-6i$

~~1+2i~~ $\bar{u} \times \bar{v} = (1-2i) \times (2-2i)$

$= -2-6i$

So $-2-6i = -2-6i$.

$\overline{uv} = \bar{u} \times \bar{v}$ LHS = RHS

(e) u and v are two complex numbers, such that $|u+v|^2 = |u-v|^2$.

Prove that $u\bar{v}$ is purely imaginary.

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M5

Extra paper if required.
Write the question number(s) if applicable.

QUESTION NUMBER

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Annotated Exemplar Template

Merit exemplar 2016

Subject:	Calculus	Standard:	91577	Total score:	15

Q	Grade score	Annotation
1	M5	This question provides evidence for M5 because the candidate has gained 1 r grade for their efforts in part d) a) The position of s on the Argand diagram is clearly identified. b) The candidate has correctly found A by substituting f(-3) = 16. c) The candidate has made a simple algebraic error, dropping the negative sign in front of p. d) The candidate has the correct solution and there is sufficient evidence of algebraic manipulation. e) The candidate has made an algebraic error by putting both expressions on the same side of the equation.
2	M5	This question provides evidence for M5 because the candidate has gained 1 r grade for their efforts in part d) a) The candidate has the correct expression for $a \pm \sqrt{b}i$, although the first equation on the last line is not correct. b) The working is correct but there is no statement of m. c) The candidate has not made sufficient progress to be awarded a u grade. d) The complex expression has been simplified but and the modulus has been correctly calculated, a minor error in the working, a missing “+” sign has been ignored. e) The quadratic has not been expanded correctly – no grade awarded.
3	M5	This question provides evidence for M5 because the candidate has gained 1 r grade for their efforts in part c) a) The denominator has not been successfully rationalised, because $\left(\sqrt{3}\right)^2 \neq 9$ b) The complex numbers have been successfully divided in polar form, and the answer is exact. c) Correct solution using complex conjugate to find the quadratic factor. d) The candidate has found that this statement is true for this pair of complex numbers, but this in itself, is not a proof. A proof must only use general terms. e) Not attempted.