91577

3

SUPERVISOR'S USE ONLY

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2015

91577 Apply the algebra of complex numbers in solving problems

2.00 p.m. Wednesday 25 November 2015
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–11 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 24

ASSESSOR'S USE ONLY

QUESTION ONE

ASSESSOR'S USE ONLY

(a) Solve the equation $x^2 - 8x + 4 = 0$.

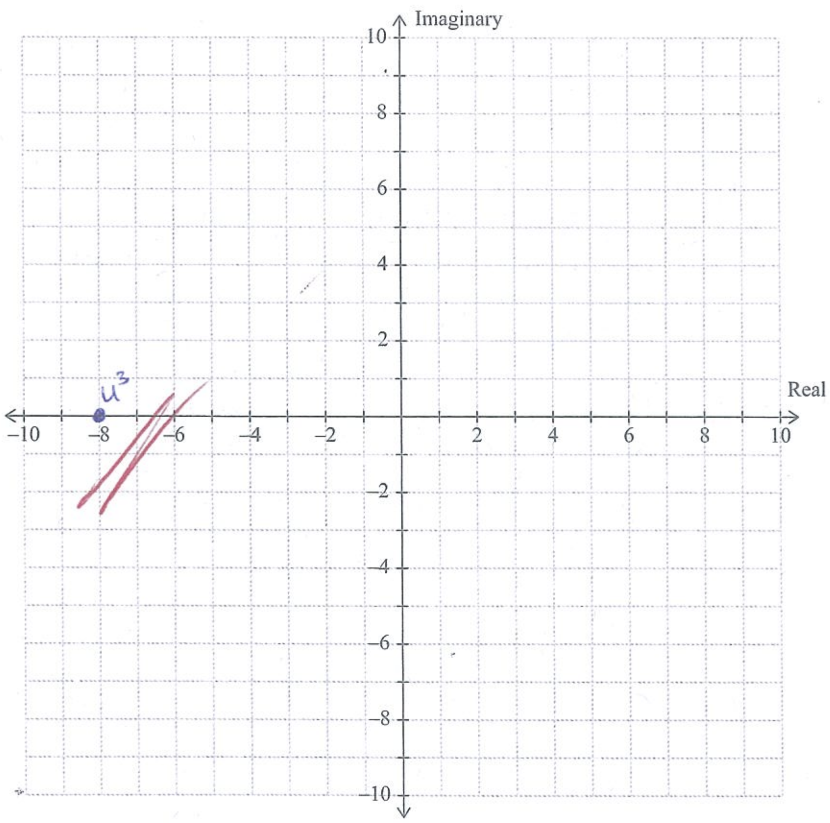
Write your answer in the form $a \pm b\sqrt{c}$, where a, b, and c are integers and $b \neq 1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{8 \pm \sqrt{64-16}}{2} = \frac{8 \pm \sqrt{48}}{2} = \frac{8 \pm 4\sqrt{3}}{2} = 4 \pm 2\sqrt{3}$$

U

(b) If $u = 1 + \sqrt{3}\mathrm{i}$, clearly show u^3 on the Argand diagram below.



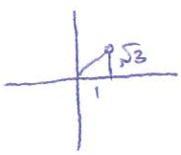
U

$r = \sqrt{1+3} = \sqrt{4} = 2$

$1+3 = \sqrt{4} = 2$

$r \operatorname{cis} \theta$

$u = 1 + \sqrt{3}i$



$\theta =$ $\sqrt{3}$

$\tan\theta =$

$\theta = \pi/3$

$2 \operatorname{cis} \pi/3$

$(2 \operatorname{cis} \pi/3)^3 = 8 \operatorname{cis} \pi = -8$

(c) v is the complex number $3 - 7i$
w is the complex number $-4 + 6i$.

Find the real numbers p and q such that $pv + qw = 6.5 - 11i$.

ASSESSOR'S USE ONLY

$V = 3-7i$

$W = -4+6i$

$p(3-7i) + q(-4+6i) = 6.5 - 11i$

$3p - 7pi - 4q + 6qi = 6.5 - 11i$

$(3p-4q) + (6q-7p)i = 6.5 - 11i$

$3p - 4q = 6.5 \quad \ldots ①$

$-7p + 6q = -11 \quad \ldots ②$

$① \times 3 = \quad 9p - 12q = 19.5$

$② \times 2 = \quad -14p + 12q = -22$

$= -5p = -2.5$

$p = \frac{2.5}{5} = 0.5$

$p = 0.5, \quad 1.5 - 4q = 6.5, \quad q = -1.25$

$p = 0.5, q = -1.25$

r

(d) Prove that the roots of the equation $3x^2 + (2c + 1)x - (c + 3) = 0$ are always real for all values of c, where c is real.

$$x = \frac{-(2c+1) \pm \sqrt{(2c+1)^2 + 4(c+3)(3)}}{6}$$

$$= \frac{-2c-2 \pm \sqrt{4c^2+4c+1+(4c+12)3}}{6}$$

$$= \frac{-2c-2 \pm \sqrt{4c^2+16c+37}}{6}$$

c is always real, so $\sqrt{4c^2+16c+37}$ is also real, so no imaginary roots.

u

ns

(e) If $x^2 + bx + c$ and $x^2 + dx + e$ have a common factor of $(x - p)$,

prove that $\dfrac{e-c}{b-d} = p$, where b, c, d, e, and p are all real.

$(x-p)$ is factor of x^2+bx+c & x^2+dx+e

$f(x) = x^2+bx+c$ $\quad$ $g(x) = x^2+dx+e$

$f(p) = 0$ $\quad$ $g(\cancel{x}p) = 0$

$p^2+pb+c = 0$ $\quad$ $p^2+pd+e = 0$

$$p^2+pb+c = p^2+pd+e$$

$$pb+c = pd+e$$

$$pb-pd = e-c$$

$$p(b-d) = e-c$$

$$p = \frac{e-c}{b-d}$$ Q.E.D

ASSESSOR'S USE ONLY

t

E8

QUESTION TWO

ASSESSOR'S USE ONLY

(a) What is the remainder when $2x^3 + x^2 - 5x + 7$ is divided by $x + 3$?

$p(x) = 2x^3 + x^2 - 5x + 7$

$p(-3) =$ Remainder.

$p(-3) = -54 + 9 + 15 + 7 = 0$

$= -23.$

u

(b) The complex number $\frac{2+3\text{i}}{5+\text{i}}$ can be expressed in the form $k(1 + \text{i})$, where k is a real number.

Find the value of k.

$\frac{(2+3i)(5-i)}{(5+i)(5-i)} = \frac{10-2i+15i-3i^2}{25+1}$

$= \frac{13+13i}{26}$

$\frac{13}{26}(1+i) = \frac{1}{2}(1+i)$

$k = 1/2.$

u

* (c) Find real numbers A, B and C such that $\dfrac{1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x-1)}$

ASSESSOR'S USE ONLY

$$\frac{Ax^2(x-1) + Bx(x-1) + Cx^3}{x^3(x-1)}$$

$$\frac{Ax^3 - Ax^2 + Bx^2 - Bx + Cx^3}{x^3(x-1)} = \frac{Ax^2 - Ax + Bx - B + Cx^2}{x^2(x-1)}$$

$$Ax^2 - Ax + Bx - B + Cx^2 = 1$$

$$x^2(A+C) + x(B-A) - B - 1 = 0$$

u

ns

(d) Write the complex number $\left(\dfrac{4i^7 - i}{1+2i}\right)^2$ in the form $a + bi$, where a and b are real numbers.

$i^7 = -i$ $\qquad \left(\dfrac{-4i-i}{1+2i}\right)^2 = \left(\dfrac{-5i}{1+2i}\right)^2$

$$\frac{-5i}{1+2i} = \frac{-5i(1-2i)}{(1+2i)(1-2i)} = \frac{-5i+10i^2}{1-4i^2} = \frac{-10-5i}{5}$$

$$\left(\frac{-5i}{1+2i}\right)^2 = \left(\frac{-10-5i}{5}\right)^2 = \frac{(-10-5i)^2}{25} = \frac{100+100i-25}{25}$$

$$= \frac{75+100i}{25}$$

$$= 3+4i$$

r

$$\frac{1}{x^2(x-1)} = \frac{\cancel{A}}{x} \quad \frac{1}{x^3 - x^2}$$

ASSESSOR'S USE ONLY

(e) Find the Cartesian equation of the locus described by $\arg\left(\frac{z-2}{z+5}\right) = \frac{\pi}{4}$

let $z = x + yi$

$$= \frac{(x-2)+yi}{(x+5)+yi} = \frac{(x-2+yi)(x+5-yi)}{(x+5+yi)(x+5-yi)}$$

$$= \cancel{\frac{x^2+5x-xyi-2x-10+2yi+5yi+xyi+y^2}{x^2+5x-xyi+5x+25-5yi+xyi+5yi+y^2}}$$

$$= \frac{x^2+y^2+3x+\cancel{5y}-10+\cancel{2yi}\;7yi}{x^2+y^2+10x+25}$$

$$\arg\left(\frac{x^2+3x+y^2-10}{x^2+y^2+10x+25} + \frac{7yi}{x^2+y^2+10x+25}\right) = \frac{\pi}{4}$$

$$\frac{\text{Imaginary}}{\text{Real}} = \tan\frac{\pi}{4} = 1.$$

$$\frac{7y}{x^2+3x+y^2-10} = 1$$

$\arg(1+i) = \frac{\pi}{4}$

$\tan\theta = \frac{1}{1}$

$\theta = \pi/4$

$\frac{1}{1} = \frac{\pi}{4}$ 1

$$7y = x^2+3x+y^2-10$$

$$x^2+3x+y^2-7y = 10$$

$$(x^2+3x+2.25)+y^2-7y = 12.25$$

$$(x+1.5)^2+(y^2-7y+12.25) = 24.5$$

$$(x+1.5)^2+(y-3.5)^2 = 24.50$$

Ellipse

$$\frac{(x+1.5)^2}{24.5}+\frac{(y-3.5)^2}{24.5} = 1$$

t

E8

QUESTION THREE

ASSESSOR'S USE ONLY

(a) If $z = 4 + 2i$ and $w = -1 + 3i$, find $\arg(zw)$.

$zw = (4+2i)(-1+3i)$

$= -4+12i-2i+6i^2$

$= -4-6+10i$

$= -10+10i$

$\tan\alpha = -1$ $\quad \alpha = -\pi/4$ argument

(b) For what real value(s) of k does the equation $kx^2 + \frac{x}{k} + 2 = 0$ have equal roots?

$b^2 - 4ac = 0.$

$\left(\frac{1}{k}\right)^2 - 4(2k) = 0$

$\frac{1}{k^2} - 8k = 0$

$\frac{1}{k^2} = 8k$

$1 = 8k^3$

$\frac{1}{8} = k^3$, $\quad k = 1/2$

(c) One solution of the equation $3w^3 + Aw^2 - 3w + 10 = 0$ is $w = -2$.

If A is a real number, find the value of A and the other two solutions of the equation.

$W = -2$ is solutic.

$$\begin{array}{r|l} & 3w^2 + (A-6)w + (9-2A) \\ w+2 & 3w^3 + Aw^2 - 3w + 10 \\ & 3w^3 + 6w^2 \\ \hline & Aw^2 - 6w^2 - 3w + 10 \\ & Aw^2 - 6w^2 + 2Aw - 12w \\ \hline & -3w - 2Aw + 12w + 10 \\ & = -2Aw + 9w + 10 \\ & w(9-2A) + 10 \\ & w(9-2A) + 2(18-4A) \\ \hline & 10 - 18 + 4A \\ & = -8 + 4A \end{array}$$

$4A - 8 = 0$

$4A = 8$

$A = 2$

$3w^2 - 4w + 5$

AT BACK

A

4

r

(d) Solve the equation $z^3 = k + \sqrt{3}\,k\mathrm{i}$, where k is real and positive.

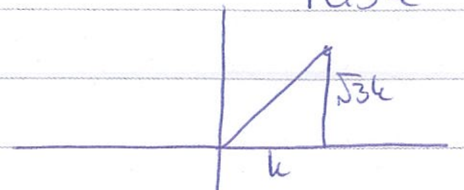
Write your solutions in polar form in terms of k.

ASSESSOR'S USE ONLY

$z^3 = k + \sqrt{3}ki$ $\qquad r\operatorname{cis}\theta$

$z^3 = 2k\operatorname{cis}(\pi/3)$

$z = (2k\operatorname{cis}\pi/3)^{1/3}$



$\cancel{z = \sqrt[3]{2k}\operatorname{cis}\frac{\pi}{9} + \frac{2k\pi}{3}}$

$\sqrt{4k^2} = 2k.$ $\qquad r = \sqrt{k^2 + 3k^2}$

~~k=0, z_1 =~~

~~k=1, z_2 =~~ $\qquad \theta = \tan^{-1}(\sqrt{3}) = \pi/3$

~~k=2, z_3 =~~

$z = \sqrt[3]{2k}\operatorname{cis}\frac{\pi}{9} + \frac{2n\pi}{3}$

1. $n = 0,\ z_1 = \sqrt[3]{2k}\operatorname{cis}\pi/9$
2. $n = 1,\ z_2 = \sqrt[3]{2k}\operatorname{cis}7\pi/9$
3. $n = 2,\ z_3 = \sqrt[3]{2k}\operatorname{cis}13\pi/9 \quad \left(\sqrt[3]{2k}\operatorname{cis}-\frac{5\pi}{9}\right)$

Question Three continues on the following page.

(e) (i) Find each of the roots of the equation $z^5 - 1 = 0$.

$z^5 = 1\text{cis}\,0$

$z = 1\text{cis}\,\frac{2k\pi}{5}$

$k=0,\ z_1 = 1\text{cis}\,0 = 1$

$k=1,\ z_2 = 1\text{cis}\,2\pi/5$

$k=2,\ z_3 = 1\text{cis}\,4\pi/5$

$k=3,\ z_4 = 1\text{cis}\,6\pi/5$

$k=4,\ z_5 = 1\text{cis}\,8\pi/5$

(ii) Let p be the root in part (i) with the smallest positive argument.

Show that the roots in part (i) can be written as $1, p, p^2, p^3, p^4$.

Smallest positive argument $= 2\pi/5$. $(1\text{cis}\,2\pi/5)$

$p = 1\text{cis}\,2\pi/5$

Roots are $= 1,\ 1\text{cis}\,2\pi/5,\ 1\text{cis}\,4\pi/5,\ 1\text{cis}\,6\pi/5,\ 1\text{cis}\,8\pi/5.$

De moivres theorem $=$ $(1\text{cis}\,2\pi/5)^2 = 1\text{cis}\,4\pi/5$

$p^2 = 1\text{cis}\,4\pi/5$

$p^3 = (1\text{cis}\,2\pi/5)^3 = 1\text{cis}\,6\pi/5$

$p^4 = (1\text{cis}\,2\pi/5)^4 = 1\text{cis}\,8\pi/5$

So, roots can be written as: $1, p, p^2, p^3, p^4$.

ASSESSOR'S USE ONLY

t

E8

Extra paper if required.
Write the question number(s) if applicable.

QUESTION NUMBER

ASSESSOR'S USE ONLY

3e) $f(w) = 3w^3 + Aw^2 - 3w + 10 = 0$. $\quad w = -2$ is solution

$f(-2) = 0$

$= -24 + 4A + 6 + 10 = 0$

$4A - 8 = 0$

$4A = 8$

$A = 2$.

$f(w) = 3w^3 + 2w^2 - 3w + 10$.

$$\begin{array}{r} 3w^2 - 4w + 5 \\ w+2 \overline{) 3w^3 + 2w^2 - 3w + 10} \\ \underline{3w^3 + 6w^2} \\ -4w^2 - 3w + 10 \\ \underline{-4w^2 - 8w} \\ 5w + 10 \\ \underline{5w + 10} \\ 0 \end{array}$$

$(w+2)(3w^2 - 4w + 5)$

$3w^2 - 4w + 5 = 0$

$w = \dfrac{4 \pm \sqrt{16 - 60}}{6}$

$= \dfrac{4 \pm \sqrt{44}\, i}{6}$

$\dfrac{4 \pm 2\sqrt{11}\, i}{6} = \dfrac{2 \pm \sqrt{11}\, i}{3}$

So, $A = 2$

Other 2 solutions are:

$w = \dfrac{2 \pm \sqrt{11}\, i}{3}$.

graded p8

Excellence exemplar for 91577 2015		**Total score**	**24**
Q	**Grade score**	**Annotation**	
1	E8	This is an E8 because the proof required in part e) was completed correctly with clear communication of the candidate's understanding of the factor theorem shown. a) The candidate has substituted into the quadratic formula and then written their solution in its most simplified form. b) The complex number has been converted from rectangular form to polar form. Then the candidate has used De Moivre's Theorem to raise the complex number to the power of 3. They have also indicated its position on the Argand diagram. c) The correct pair of equations in two unknowns has been found by substituting for v and w into the given equation and then equating the real and imaginary parts. The candidate has solved the two equations simultaneously to find the values of p and q. d) The candidate has demonstrated that they can find the discriminant of the given quadratic equation but has shown that they do not understand how they can use it to prove that the roots of the equation will always be real. e) The candidate has set up the two corresponding functions for the given quadratic expressions and demonstrated their understanding of the factor theorem by stating that if $(x-p)$ is a factor then f(p) and g(p) would both result in a zero remainder. The proof follows quickly after equating the two equations resulting from f(p) and g(p) that both equal zero.	
2	E8	This is an E8 because the Cartesian equation required in part e) was completed correctly. a) The remainder theorem was correctly applied. b) The quotient of the two complex numbers written in rectangular form was simplified correctly to arrive at a rational denominator and the k value was identified. c) The three algebraic fractions were correctly written over a common denominator. The candidate equated the numerators of each side of the equation which is why they scored a u. In order to gain the r grade, they needed to continue to compare coefficients of the expressions on each side of the equation. d) The candidate successfully simplified all the terms involving i in the numerator then was able to square the resulting complex number. e) The candidate has substituted for z in both the numerator and the denominator then multiplied top and bottom by the conjugate of the denominator in order to identify the real and imaginary parts of the simplified complex number. The candidate has realised that if the argument of the complex number is $\frac{\pi}{4}$, then the real and imaginary parts must be equal and has used this piece of information to arrive at the required equation of the circle.	
3	E8	This is an E8 because the candidate has demonstrated an understanding of de Moivre's Theorem in part e to both find the required roots of the quintic equation and then demonstrate the required relationships between the roots. a) The product of zw was found but not the correct argument. The diagram is accurate but the angle measured anticlockwise from the positive direction of the real axis has not been identified. b) The candidate realised that the discriminant had to equal zero if the roots of the given quadratic equation were equal and has solved the resulting equation. c) The candidate has divided the cubic by the factor corresponding to the root provided and has subsequently found the value of the unknown coefficient, A, as well as the quadratic factor of the cubic equation. They have then used the quadratic formula to find the other two roots of the cubic. d) The complex number has been converted to its polar form and the 3 required cube roots found. e) The complex number has been converted to its polar form and the 5 required roots found. The root with the smallest positive argument has been identified and by using de Moivre's Theorem, the candidate has shown how 3 of the roots can be expressed as the required powers of p.	