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NZQA
NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2015

91577 Apply the algebra of complex numbers in solving problems

2.00 p.m. Wednesday 25 November 2015
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–11 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Achievement

TOTAL 10

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QUESTION ONE

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(a) Solve the equation $x^2 - 8x + 4 = 0$.

Write your answer in the form $a \pm b\sqrt{c}$, where a, b, and c are integers and $b \neq 1$.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad x = \dfrac{8 \pm \sqrt{64 - 4 \times 1 \times 4}}{2 \times 1}$

$x = \cancel{8} \pm 4 \pm \frac{1}{2}\sqrt{48}$

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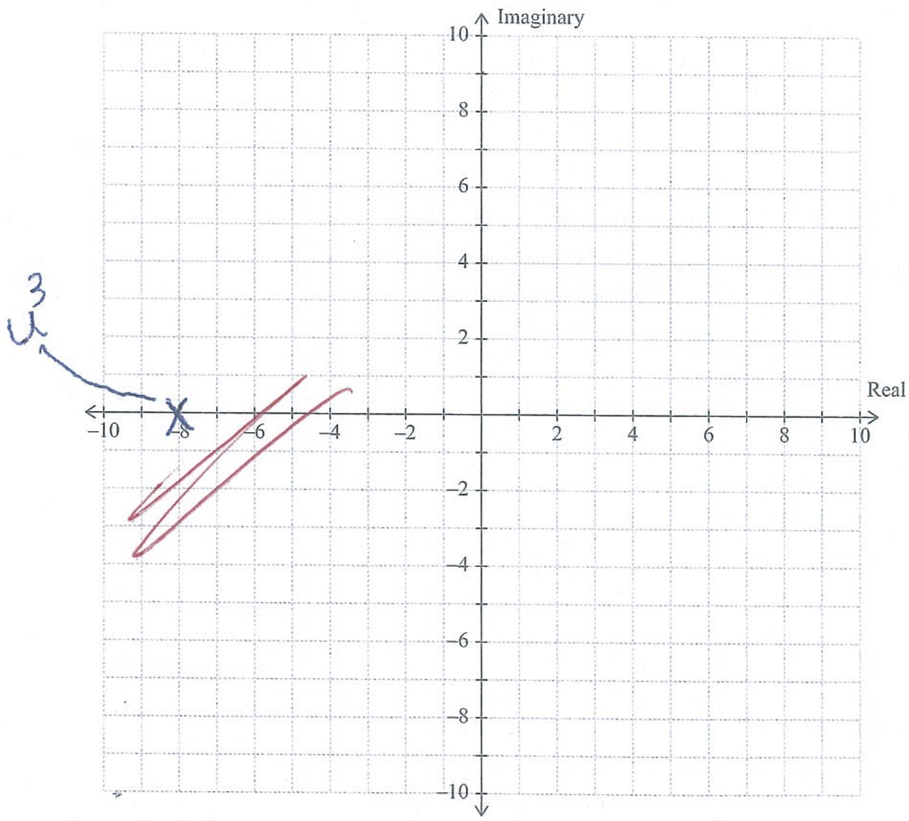
(b) If $u = 1 + \sqrt{3}\text{i}$, clearly show u^3 on the Argand diagram below.



U

$1 + \sqrt{3}i$

$u\,\cancel{x} = 2cis(1.047)$

$\cancel{x}\,u^3 = 8cis(\pi)$

Rec : $-8 + 0i$

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(c) v is the complex number $3 - 7i$
w is the complex number $-4 + 6i$.

Find the real numbers p and q such that $pv + qw = 6.5 - 11i$.

$3p - 7ip \cancel{w} - 4q + 6iq = 6.5 - 11i$

$p(3 - 7i) + q(-4 + 6i) = 6.5 - 11i$

u

ns

$(2c+1)(2c+1)$

(d) Prove that the roots of the equation $3x^2 + (2c + 1)x - (c + 3) = 0$ are always real for all values of c, where c is real.

$-(2c+1) \pm \sqrt{(2c+1)^2 - 4 \times 3 \times -(c+3)}$

$\sqrt{4c^2 + 4c + 1 - 12(c+3)}$

$\sqrt{4c^2 + 4c + 1 + 12c + 36}$

$\sqrt{4c^2 + 16c + 37}$

$4c^2 + 16c + 37 > -1$ gives real answers

u

n

(e) If $x^2 + bx + c$ and $x^2 + dx + e$ have a common factor of $(x - p)$,

prove that $\dfrac{e-c}{b-d} = p$, where b, c, d, e, and p are all real.

$$\begin{array}{r}
x+(b+p) \\
x-p \,\overline{)\, x^2+bx+c} \\
x^2-px
\end{array}$$

A

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A4

QUESTION TWO

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(a) What is the remainder when $2x^3 + x^2 - 5x + 7$ is divided by $x + 3$?

$x = -3 \quad 2(-3)^3 + (-3^2) - 5 \times -3 + 7$

$= -54 \times 9 + 15 + 7$

$= -23$ remainder

u

(b) The complex number $\frac{2+3i}{5+i}$ can be expressed in the form $k(1 + i)$, where k is a real number.

Find the value of k.

$\frac{2+3i}{5+i} \cdot \frac{5-i}{5-i} = \frac{10+15i-2i-3i^2}{25+5i-5i-i^2} = \frac{13+13i}{25+1}$

$\frac{13+13i}{26} = \frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1+i)$

$k = \frac{1}{2}$

u

(c) Find real numbers A, B and C such that $\dfrac{1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x-1)}$

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$= \dfrac{1}{x^3 - x^2}$

$A = 0$ $B = \cancel{4}\,1$ $C = 1$.

n

(d) Write the complex number $\left(\dfrac{4i^7 - i}{1+2i}\right)^2$ in the form $a + bi$, where a and b are real numbers.

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(e) Find the Cartesian equation of the locus described by $\arg\left(\frac{z-2}{z+5}\right)=\frac{\pi}{4}$

let $z = x+iy$ $\quad \frac{x+iy-2}{x+iy+5} = \frac{\pi}{4}$

$\frac{(x-2)^2+i(y)^2}{(x+5)^2+i(y)^2} = \frac{\pi}{4}$ $\quad \frac{\pi}{4}\times\frac{\pi}{4}$ $\quad \frac{\pi^2}{4^2}$

$4^2(x^2-4x+4+y^2) = \pi^2(x^2+10x+25+y^2)$

$$\frac{x+iy-2}{x+iy+5} = \frac{\pi}{4}$$

$$\frac{(x-2)^2+y^2}{(x+5)^2+(y)^2}$$

$$\frac{(x-2)^2+y^2}{(x+5)^2+y^2} = \frac{\pi}{4}$$

$$\frac{x^2-4x+4+y^2}{x^2+10x+20+y^2} = \frac{\pi}{4}$$

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A3

QUESTION THREE

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(a) If $z = 4 + 2\text{i}$ and $w = -1 + 3\text{i}$, find $\arg(zw)$.

$(4+2i)(-1+3i)$

$-4-2i+12i+6i^2$ $= 2.35$

$= -10+10i$

$\arg(zw) = \frac{3\pi}{4}$

u

(b) For what real value(s) of k does the equation $kx^2 + \frac{x}{k} + 2 = 0$ have equal roots?

$\sqrt{(\frac{1}{k})^2 - 8k}$

$\frac{1}{k^2} - 8k = 0$ $\quad \frac{1}{k^2} = 8k$ $\quad 8k^3 = 1$

$k^3 = \frac{1}{8}$

$k = 0.5$

u

(c) One solution of the equation $3w^3 + Aw^2 - 3w + 10 = 0$ is $w = -2$.

If A is a real number, find the value of A and the other two solutions of the equation.

$(w+2)$ is a factor

$w+2 \overline{) 3w^3 + Aw^2 - 3w + 10}$ quotient: $3w^2 + (A-6w)w + (-3-2a-12w)$

$3w^3 + 6w^2$

$2a - 12w$

$3w^2 + (A-6)w + (-3-2a+12w)$

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(d) Solve the equation $z^3 = k + \sqrt{3}\,k\mathrm{i}$, where k is real and positive.

Write your solutions in polar form in terms of k.

2 , 1.0471

$z^3 = 2k \operatorname{cis}\left(\frac{\pi}{3}\right)$

$z = 2k^{1/3} \operatorname{cis}\left(\frac{\frac{\pi}{3} + 2k\pi}{3}\right)$

$z^0 = 2\sqrt[3]{k} \operatorname{cis}\left(\frac{\pi}{6}k\right)$

$z^1 = 2\sqrt[3]{k} \operatorname{cis}\left(\frac{\frac{\pi}{3}k + 2\pi}{3}\right) = 2\sqrt[3]{k} \operatorname{cis}\frac{2\pi}{3}k$

$z^2 = 2\sqrt[3]{k} \operatorname{cis}\left(\frac{\frac{\pi}{3}k + 4\pi}{3}\right) = 2\sqrt[3]{k} \operatorname{cis}\frac{13\pi}{6}k$

$z^0 = 2\sqrt[3]{k} \operatorname{cis}\left(\frac{\pi}{6}k\right)$

$z^1 = 2\sqrt[3]{k} \operatorname{cis}\left(\frac{2\pi}{3}k\right)$

$z^2 = 2\sqrt[3]{k} \operatorname{cis}\left(\frac{7\pi}{6}k\right)$

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Question Three continues on the following page.

(e) (i) Find each of the roots of the equation $z^5 - 1 = 0$.

$z^5 = 1 \quad z^5 = 1\text{cis}(0)$

$z^0 = 0$

$z^1 = \text{cis}\left(\frac{\pi}{5}\right)$

$z^2 = \text{cis}\left(\frac{2\pi}{5}\right)$

$z^3 = \text{cis}\left(\frac{3\pi}{5}\right)$

$z^4 = \text{cis}\left(\frac{4\pi}{5}\right)$

$z^5 = \text{cis}(\pi)$

(ii) Let p be the root in part (i) with the smallest positive argument.

Show that the roots in part (i) can be written as $1, p, p^2, p^3, p^4$.

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A3

Achieved exemplar for 91577 2015			Total score	10
Q	**Grade score**	**Annotation**		
1	A4	This question provides evidence for A4 because the candidate has gained 3 u grades for their efforts in parts b), c) and d) a) The solution for the quadratic equation is correct but not sufficient because it has not been expressed in its most simplified surd form. b) u^3 has been found and indicated on the Argand diagram. c) The candidate has substituted for v and w and expanded the brackets to simplify the left-hand side of the equation. The candidate has not gone on to equate the real and imaginary parts or solve the resulting pair of equations in p and q. d) The discriminant has been found. No progress has been made to show that it must always be positive. e) The candidate has made a start by dividing the given factor into the first quadratic expression.		
2	A3	This question provides evidence for A3 because the candidate has correctly completed the two achievement level questions a) and b). a) The remainder has been used to find the required -23. b) The quotient of the two complex numbers has been simplified to find the required $k = \frac{1}{2}$ c) The candidate has provided a correct value for C but incorrect values for A and B without any supporting working. d) Not attempted. e) The candidate has substituted for z but has confused the modulus with the argument.		
3	A3	This question provides evidence for A3 because the candidate has correctly completed the two achievement level questions a) and b). a) The candidate has correctly multiplied the complex numbers, z with w. There is no working to support the correct answer for the argument. Most likely the student has used the arg option on their graphic calculator. b) The discriminant has been found and solved equal to zero. c) An algebraic long division involving the factor corresponding to the given solution has been unsuccessfully attempted. d) The complex number has been converted to polar form but the following attempts to use de Moivre's Theorem to find the required roots have been unsuccessful. e) i) As with part d), the candidate has converted from rectangular form to polar form but has not been able to complete the problem. e) ii) Not attempted.		