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Level 3 Calculus, 2017

91577 Apply the algebra of complex numbers in solving problems

9.30 a.m. Thursday 23 November 2017
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–12 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Merit

TOTAL 15

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QUESTION ONE

(a) If $u = 2 + 3i$ and $v = 1 - 4i$, find $\bar{u} - 3v$, giving your solution in the form $a + bi$.

$= 2 - 3i - 3(1 - 4i)$ $\qquad \bar{u} = 2 - 3i$

$= 2 - 3i - 3 + 12i$

$= -1 + 9i$

(b) Write $\frac{36}{5-\sqrt{7}}$ in the form $a + b\sqrt{7}$, where a and b are integers.

$= \frac{36(5+\sqrt{7})}{(5-\sqrt{7})(5+\sqrt{7})}$

$= \frac{180 + 36\sqrt{7}}{25 + 5\sqrt{7} - 5\sqrt{7} - 7}$

$= \frac{180 + 36\sqrt{7}}{18}$

$= 10 + 2\sqrt{7}$

(c) Solve the following equation for x in terms of p:

$$p\sqrt{x-2} - 5\sqrt{x} = 0$$

$p^2 = \frac{25x}{(x-2)}$

$(p\sqrt{x-2})^2 = (5\sqrt{x})^2$

$p^2(x-2) = 25x$

$\frac{p^2x - 2p^2}{p^2} = \frac{25x}{p^2}$

$x - 2 = \frac{25x}{p^2}$

$x = \frac{25x}{p^2} + 2$

$\cancel{p^2x - 25x - 2p^2 = 0}$

$p^2(x-2) \cancel{- 25x} = \cancel{25x}$

$\cancel{x}$

$(kx+c)$ $(-\sqrt{i})^2$

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(d) One solution of the equation $z^3 - 2z^2 + Bz - 30 = 0$ is $z = -2 - i$.

$z = -2 + i$

If B is a real number, find the value of B and the other two solutions of the equation.

$z_1 = -2 - i$ as B is real, $z_2 = -2 + i$

$(z - (-2 - i))(z - (-2 + i))$

$= (z + 2 + i)(z + 2 - i)$

$= z^2 + 2z - iz + 2z + 4 - \cancel{2i} + iz + \cancel{2i} + 1$

$= z^2 + 4z + 5$

$\therefore z^3 - 2z^2 + Bz - 30 = (z^2 + 4z + 5)(kz + c)$

$z^3 = z^2 \times kz$ so $k = 1$

$-30 = 5 \times c$ so $c = -6$ $z_3 = 6$

$(z^2 + 4z + 5)(z - 6) = 0$

$z^3 + 4z^2 + 5z - 6z^2 - 24z - 30 = 0$

$z^3 - 2z^2 - 19z - 30 = 0$

$Bz = -19z$

so $B = -19$ and the other two solutions are: $z = -2 + i$ and $z = 6$

(e) Find the Cartesian equation of the locus described by $|z + 2 - 7i| = 2|z - 10 + 2i|$.

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Write your answer in the form $(x + A)^2 + (y + B)^2 = K$.

let $z = x + iy$

$|x + iy + 2 - 7i| = 2|x + iy - 10 + 2i|$

$|x + 2 + i(y-7)| = 2|x - 10 + i(y+2)|$

$\sqrt{(x+2)^2 + (y-7)^2} = \sqrt{(x-10)^2 + (y+2)^2}$

$\left(\sqrt{x^2 + 4x + 4 + y^2 - 14y + 49}\right)^2 = \left(\sqrt{x^2 - 20x + 100 + y^2 + 4y + 4}\right)^2$

$\cancel{x^2} + 4x + \cancel{y^2} - 14y + 53 = \cancel{x^2} + \cancel{y^2} - 20x + 4y + 104$

$4x - 14y + 53 = -20x + 4y + 104$

$24x - 18y - 51 = 0$

$24x - 18y = 51$

$y = \frac{-24x + 51}{-18}$

$y = \frac{4}{3}x + \frac{-17}{6}$

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QUESTION TWO

(a) Dividing $x^3 - 2x^2 + 5x + d$ by $(x - 3)$ gives a remainder of 13.

Find the value of d.

$$x^2 + x + 8$$
$$(x-3)\overline{)x^3 - 2x^2 + 5x + d}$$
$$\ominus\ x^3 - 3x^2$$
$$x^2 + 5x$$
$$\ominus\ x^2 - 3x$$
$$8x + d$$
$$\ominus\ 8x - 24$$
$$13$$

$d + 24 = 13$

$d = -11$

(b) Simplify, as far as possible, the expression $\sqrt{2k}^2\left(\sqrt{18k} - \sqrt{8k}\right)^2$.

$= \sqrt{2k}^2\left(\sqrt{18k} - \sqrt{8k}\right)^2$

$= 2k\left(18k - 2\left(\sqrt{8k} \times \sqrt{18k}\right) + 8k\right)$

$= 2k\left(26k - 2(12k)\right)$

$= 2k(26k - 24k)$

$= 2k \times 2k$

$= 4k^2$

(c) z and w are complex numbers such that $z = -2 + 3i$ and $zw = 15 - 3i$.

Find an exact value of $\arg(w)$.

$\frac{\pi}{2} + \frac{\pi}{4}$

$= \frac{2\pi}{4} + \frac{\pi}{4}$

$w(-2 + 3i) = 15 - 3i$

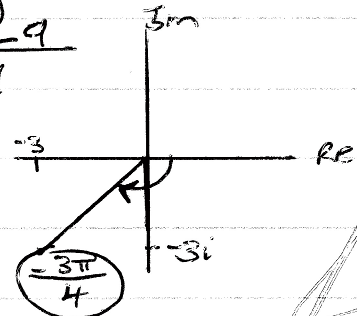
$w = \frac{15 - 3i}{-2 + 3i}$

$w = \frac{(15 - 3i)(-2 - 3i)}{(-2 + 3i)(-2 - 3i)}$

$= \frac{-30 - 45i + 6i - 9}{4 + 6i - 6i + 9}$

$= \frac{-39 - 39i}{13}$

$= -3 - 3i$



$\arg(w)$ is at $(-3, -3)$

$= -\frac{3\pi}{4}$

z^4 so add $\frac{\pi}{2}$ to each

$\frac{2\pi}{4}$ $\frac{\pi}{2}$

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(d) Solve the equation $z^4 = \frac{m}{\sqrt{2}} + \frac{m}{\sqrt{2}}\mathrm{i}$, where m is real and positive.

$r \operatorname{cis} \theta$

Write your solutions in polar form in terms of m.

$\frac{m}{\sqrt{2}} + \frac{m}{\sqrt{2}}i$ in polar form =

$r = \sqrt{\left(\frac{m}{\sqrt{2}}\right)^2 + \left(\frac{m}{\sqrt{2}}\right)^2}$

$= \sqrt{\frac{m}{\sqrt{2}} \times \frac{m}{\sqrt{2}} + \frac{m}{\sqrt{2}} \times \frac{m}{\sqrt{2}}}$

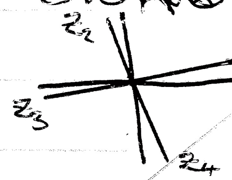
$= \frac{m^2}{2} + \frac{m^2}{2}$

$= 2\frac{m^2}{2} = m^2$

$\theta = \tan^{-1}\left(\frac{\frac{m}{\sqrt{2}}}{\frac{m}{\sqrt{2}}}\right)$

$= \tan^{-1}(1) = 45° = \frac{\pi}{4}$

$z^4 = m^2 \operatorname{cis} \frac{\pi}{4}$



$z_1 = m^{\frac{1}{2}} \operatorname{cis} 2.59°$

$z_2 = m^{\frac{1}{2}} \operatorname{cis} 92.59°$

$z_3 = m^{\frac{1}{2}} \operatorname{cis} -177.4°$

$z_4 = m^{\frac{1}{2}} \operatorname{cis} -87.41$

(e) Find all possible values of k that make $u = \dfrac{k+4\mathrm{i}}{1+k\mathrm{i}}$ a purely real number.

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MS

QUESTION THREE

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(a) If $u = p^3 \text{cis}\frac{\pi}{3}$ and $v = p\,\text{cis}\frac{\pi}{8}$, write $\frac{u}{v}$ in polar form.

cis

$\frac{u}{V} = \frac{p^3 \text{cis}\frac{\pi}{3}}{p\,\text{cis}\frac{\pi}{8}}$

$= \frac{p^2 \text{cis}\frac{\pi}{3}}{\frac{\pi}{8}}$

(b) Solve the equation $x^2 - 6x + 14 = 0$.

Give your solution in the form $a \pm \sqrt{b}\,\mathrm{i}$, where a and b are rational numbers.

using QF $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 4 \times 14}}{2}$

$= \frac{6 \pm \sqrt{-20}}{2}$

$= 3 \pm \frac{\sqrt{20}}{2} i = 3 \pm \frac{\sqrt{4} \times \sqrt{5}}{2} i$

$= 3 \pm \sqrt{5}\, i$

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(c) $\dfrac{3x^3+8x^2-2x+11}{x+2} = 3x^2 + Ax + B + \dfrac{C}{x+2}$, where A, B, and C are integers.

Find the values of A, B, and C.

$$
\begin{array}{r}
3x^2+2x-6 \\
x+2\,\overline{)\,3x^3+8x^2-2x+11} \\
\ominus\, \underline{3x^3+6x^2} \\
2x^2-2x \\
\ominus\, \underline{2x^2+4x} \\
-6x+11 \\
\ominus\, \underline{-6x-12} \\
23
\end{array}
$$

$3x^2 + Ax + B + \dfrac{C}{x+2} = 3x^2 + 2x - 6 + \dfrac{23}{x+2}$

$\therefore A = 2$

$B = -6$

$C = 23$

(d) Solve the equation $\dfrac{8+x}{x} = \sqrt{3}$, writing your solution in the form $x = a + b\sqrt{3}$.

$\dfrac{8+x}{x} = \sqrt{3}$

$\cancel{8 + x = \sqrt{3}x}$

$\cancel{x =}$

$\dfrac{8}{x} + \dfrac{x}{x} = \sqrt{3}$

$\dfrac{8}{x} + 1 = \sqrt{3}$

$\dfrac{8}{x} = \sqrt{3} + 1$

$8 = x(\sqrt{3} - 1)$

$0 = x\sqrt{3} - x - 8$

x

Question Three continues on the following page.

(e) z is a complex number such that $z = \frac{a+bi}{a-bi}$, where a and b are real numbers.

Prove that $\frac{z^2+1}{2z} = \frac{a^2-b^2}{a^2+b^2}$.

$$z = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)}$$
$$= \frac{a^2+2bi+b^2}{a^2+b^2}$$
$$= 2bi-1$$

$$LHS = \frac{(2bi-1)^2+1}{2(2bi-1)}$$
$$= \frac{(2bi-1)(2bi-1)+1}{4bi-2}$$
$$= \frac{-4b^2+1}{4bi-2}$$
$$= \frac{(-4b^2+1)(-4bi+2)}{(4bi-2)(-4bi-2)}$$
$$= \frac{16b^3i+8b^2i-4bi-2}{16b^2-8b^2b+8bi+4}$$
$$= \frac{16b^3i+8b^2i-4bi-2}{16b^2+4}$$
$$= \frac{i(16b^3+8b^2-4b)-2}{16b^2+4}$$

Merit exemplar 2017

Subject:	Calculus	Standard:	91577	Total score:	15

Q	Grade score	Annotation
1	M5	This question provides evidence for M5 because the candidate has gained 1 r grade for their efforts in part d) a) The candidate has found $\bar{u} - 3v$ b) The candidate has correctly found an expression in the form, $a + b\sqrt{7}$ by rationalising the denominator. c) The candidate has not given x in terms of p, as the right hand side of the expression contains x. d) The candidate has the correct solution and there is sufficient evidence of algebraic manipulation. e) The candidate has made an early algebraic error by dropping the factor of 2.
2	M5	This question provides evidence for M5 because the candidate has gained 1 r grade for their efforts in part c) a) The candidate has correctly used long division to find d, although remainder theorem would have led to the required solution more quickly. b) The candidate has not found an equivalent expression to that given because they initially, incorrectly chose to square the expression. c) The candidate has correctly found the real and imaginary parts of w, and given the argument as required. d) The candidate has correctly found the argument of z^4 but not the modulus. The arguments given for z are incorrect. e)
3	M5	This question provides evidence for M5 because the candidate has gained 1 r grade for their efforts in part c) a) The candidate has not correctly found an expression for, $\frac{u}{v}$ because they have not subtracted the argument of v from the argument of u. b) The candidate has solved the equation and given the solution in the required form. c) The candidate has correctly used long division to create and expression equivalent to the right hand side of the equation and hence find, A, B and C. d) The candidate has not made significant progress towards a solution. e) The candidate has not made significant progress towards a solution.