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91577

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NZQA
NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2017

91577 Apply the algebra of complex numbers in solving problems

9.30 a.m. Thursday 23 November 2017
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–12 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 24

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QUESTION ONE

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(a) If $u = 2 + 3\text{i}$ and $v = 1 - 4\text{i}$, find $\bar{u} - 3v$, giving your solution in the form $a + b\text{i}$.

(b) Write $\frac{36}{5-\sqrt{7}}$ in the form $a + b\sqrt{7}$, where a and b are integers.

(c) Solve the following equation for x in terms of p:

$$p\sqrt{x-2} - 5\sqrt{x} = 0$$

$(p\sqrt{x-2})^2 = (5\sqrt{x})^2$

$p^2(x-2) = 25x$

$p^2x - 2p^2 - 25x = 0$

$x(p^2-25) = 2p^2$

$x = \frac{2p^2}{p^2-25}$

(d) One solution of the equation $z^3 - 2z^2 + Bz - 30 = 0$ is $z = -2 - i$.

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If B is a real number, find the value of B and the other two solutions of the equation.

$(-2-i)^3 - 2(-2-i)^2 + B(-2-i) - 30 = 0$

$(3+4i)$

$(4+4i-1)(-2-i) - 2(3+4i) + B(-2-i) - 30 = 0$

$(-6-3i-8i+4) - 6 - 8i + B(-2-i) - 30 = 0$

$-11i - 2 - 6 - 8i - 2B - Bi - 30 = 0$

$-19i - 38 - 2B - Bi = 0$

$-19 - B = 0$

$B = -19$

$(z+2+i)(z+2-i)$

$\boxed{z = -2-i, -2+i, 6}$

$(z+2+i)(z+2-i) = \cancel{z}^2 + 2\cancel{z} - \cancel{z}i + 2\cancel{z} + 4 - \cancel{2}i + \cancel{z}i + 2\cancel{i} + 1$

$= z^2 + 4z + 5$

$$
\begin{array}{r|l}
 & z - 6 \\
z^2+4z+5 & z^3 - 2z^2 - 19z - 30 \\
 & z^3 + 4z^2 + 5z \\
\hline
 & -6z^2 - 24z - 30 \\
 & -6z^2 - 24z - 30 \\
\hline
 & 0
\end{array}
$$

(e) Find the Cartesian equation of the locus described by $|z + 2 - 7i| = 2|z - 10 + 2i|$.

Write your answer in the form $(x + A)^2 + (y + B)^2 = K$.

~~(a)~~

$z = a + bi$

$|a + 2 + bi - 7i| = 2|a - 10 + bi + 2i|$

$\sqrt{(a+2)^2 + (b-7)^2} = 2\sqrt{(a-10)^2 + (b+2)^2}$

$a^2 + 4a + 4 + b^2 - 14b + 49 = 4(a^2 - 20a + 100 + b^2 + 4b + 4)$

$\cancel{a^2} + \cancel{4a} + \cancel{b^2} - \cancel{14b} + 53 = \cancel{4a^2} - 80a + 400 + \cancel{4b^2} + \cancel{16b} + 16$

$3a^2 - 84a + 3b^2 + 30b + 363 = 0$

$a^2 - 28a + b^2 + 10b + 121 = 0$

$(a - 14)^2 - 196 + (b + 5)^2 - 25 + 121 = 0$

$(a - 14)^2 + (b + 5)^2 = 100$

$(x - 14)^2 + (y + 5)^2 = 100$

E8

QUESTION TWO

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(a) Dividing $x^3 - 2x^2 + 5x + d$ by $(x - 3)$ gives a remainder of 13.

Find the value of d.

(b) Simplify, as far as possible, the expression $\sqrt{2k}\left(\sqrt{18k} - \sqrt{8k}\right)$.

(c) z and w are complex numbers such that $z = -2 + 3\text{i}$ and $zw = 15 - 3\text{i}$.

Find an exact value of $\arg(w)$.

$(-2+3i)(a+bi) = 15-3i$

$-2a - 2bi + 3ai - 3b = 15 - 3i$

$-2a - 3b = 15$

$3a - 2b = -3$

$\Rightarrow$

$-6a - 9b = 45$

$6a - 4b = -6$

$-13b = 39$

$\boxed{\begin{array}{c} b = -3 \\ a = -3 \end{array}}$

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(d) Solve the equation $z^4 = \frac{m}{\sqrt{2}} + \frac{m}{\sqrt{2}}\mathrm{i}$, where m is real and positive.

Write your solutions in polar form in terms of m.

(e) Find all possible values of k that make $u = \dfrac{k+4i}{1+ki}$ a purely real number.

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$$u = \frac{(k+4i)(1-ki)}{(1+ki)(1-ki)}$$

$$= \frac{k - k^2 i + 4i + 4k}{1 + k^2}$$

$$= \frac{5k - k^2 i + 4i}{1+k^2}$$

$$\frac{-k^2+4}{1+k^2} = 0$$

$$-k^2+4=0$$

$$\cancel{-k^2 = -4}$$

$$k^2-4=0$$

$$k = \pm 2$$

E8

QUESTION THREE

ASSESSOR'S USE ONLY

(a) If $u = p^3 \operatorname{cis}\frac{\pi}{3}$ and $v = p \operatorname{cis}\frac{\pi}{8}$, write $\frac{u}{v}$ in polar form.

(b) Solve the equation $x^2 - 6x + 14 = 0$.

Give your solution in the form $a \pm \sqrt{b}\,\mathrm{i}$, where a and b are rational numbers.

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(c) $\frac{3x^3+8x^2-2x+11}{x+2} = 3x^2 + Ax + B + \frac{C}{x+2}$, where A, B, and C are integers.

Find the values of A, B, and C.

(d) Solve the equation $\frac{8+x}{x} = \sqrt{3}$, writing your solution in the form $x = a + b\sqrt{3}$.

$(8+x)^2 = (\sqrt{3}\,x)^2$

$x^2+16x+64 = 3x^2$

$2x^2-16x-64=0$

$x^2-8x-32=0$

$a=1,\ b=-8,\ c=-32$

$\frac{8 \pm \sqrt{64-4\times-32}}{2} = \frac{8 \pm 8\sqrt{3}}{2}$

$= 4 \pm 4\sqrt{3}$

$2\,|\,192$
$2\,|\,96$
$2\,|\,48$
$2\,|\,24$
$2\,|\,12$
$2\,|\,6$
3

Question Three continues on the following page.

(e) z is a complex number such that $z = \frac{a+bi}{a-bi}$, where a and b are real numbers.

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Prove that $\frac{z^2+1}{2z} = \frac{a^2-b^2}{a^2+b^2}$.

$$z = \frac{(a+bi)}{(a-bi)}$$

$$\frac{z^2+1}{2z} = \frac{z}{2} + \frac{1}{2z}$$

$$= \frac{\frac{(a+bi)}{(a-bi)}}{2} + \frac{1}{2\left(\frac{a+bi}{a-bi}\right)}$$

$$= \frac{(a+bi)}{2(a-bi)} + \frac{1}{2} \times \frac{a-bi}{a+bi}$$

$$= \frac{1}{2}\left(\frac{a+bi}{a-bi} + \frac{a-bi}{a+bi}\right)$$

$$= \frac{1}{2}\left(\frac{a^2+2abi-b^2+a^2-2abi-b^2}{(a-bi)(a+bi)}\right)$$

$$= \frac{1}{2}\left(\frac{2a^2-2b^2}{a^2+b^2}\right)$$

$$= \frac{a^2-b^2}{a^2+b^2}$$

ASSESSOR'S USE ONLY

E8

Extra paper if required.
Write the question number(s) if applicable.

QUESTION NUMBER

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91577

<table>
<tr><th>Subject:</th><th>Calculus</th><th>Standard:</th><th>91577</th><th>Total score:</th><th>24</th></tr>
</table>

<table>
<tr><th>Q</th><th>Grade score</th><th>Annotation</th></tr>
<tr><td>1</td><td>E8</td><td>This question provides evidence for E8 because the candidate has gained 1 e grade for their efforts in part e)
a)
b)
c) The candidate has correctly rearranged to give x in terms of p
d) The candidate has the correct solution and there is sufficient evidence of algebraic manipulation.
e) The candidate has correctly use the general expressions for the moduli, gone on to calculate the equation of the locus and completed the square to give the equation in the required form.</td></tr>
<tr><td>2</td><td>E8</td><td>This question provides evidence for E8 because the candidate has gained 1 e grade for their efforts in part e)
a)
b)
c) The candidate has correctly found the real and imaginary parts of w, but not given the argument as required.
d)
e) u has been rearranged to make the denominator real, and the values of k which make u purely real have been found.</td></tr>
<tr><td>3</td><td>E8</td><td>This question provides evidence for E8 because the candidate has gained 1 e grade for their efforts in part e)
a)
b)
c)
d) The candidate has been awarded a u grade rather than an r grade because they have not removed the false solution created by squaring both sides of the equation.
e) The candidate has successfully completed the proof.</td></tr>
</table>