2023 NCEA Assessment Report

Subject: Calculus

Level: Level 3

Achievement standard(s): 91577, 91578, 91579

General commentary

The 2023 papers followed a similar format to previous years' papers. Candidates are encouraged to prioritise practice of past examination papers, studying and analysing the content, style, and format. Overall, candidate performance was also of a similar standard to previous years. Many candidates displayed the ability to solve problems using conventional methods, but also using innovative methods. It is disappointing to see, however, that there were also candidates who made fundamental and basic mathematical errors of various natures, which showed a lack of understanding of some aspects of their mathematics, as well as evidence of careless methods that precluded candidates reaching the desired solution.

Common pieces of advice made by panel leaders were:

- Candidates are encouraged to set out their working in a clear, logical, and systematic manner. This is a real advantage when solving the problems, providing the required evidence for the examiner as well as helping the candidate develop their strategy as they progress through the problem. This is beneficial, particularly in problems with which require an extended chain of reasoning. This will also enable the candidates to "self-check" as they progress through their solution in order to avoid the careless errors.
- Candidates must ensure that they carefully read the information provided in the question, which often provides helpful guidance and critical information, as well as helping the candidate recognise the requirement of the question. Unfortunately, many candidates were not able to be rewarded as they did not fully answer the demands and expectations of the question, e.g., evaluating the gradient at a particular point, providing only the x-coordinate when the question required the coordinates of a point.
- Candidates, whatever their target grade, are encouraged to attempt all parts of all questions. All question parts provide opportunities for success. It is not advisable for those candidates attempting to gain an overall Excellence grade to only attempt the questions that they believe to be 'excellence' questions. This strategy is not advisable as an introduced error or misinterpretation often leads to little reward. A large proportion of candidates who adopted this approach ended up with results which were well below their target grade. The examiner carefully ensures that there is sufficient time available in the examination to attempt all question parts. Candidates who have pre-planned a strategy for their success can be caught out when the more challenging questions are not familiar to them. Conversely, the weaker candidates often can gain valuable points in a question by completing part answers for the more challenging questions.
- Candidates who were not successful often attempted only one or two parts of each question. There are many opportunities to gain credit for correct working in all problems in all question parts, even though some candidates may struggle to solve the whole problem completely. Candidates are encouraged to answer at least a small portion of all question parts of the majority of paper in order to maximise their chance of success. A candidate should start their solution, perhaps not recognising the complete pathway through the question, and then often will find that the most appropriate strategy will reveal itself later in the solution.
- Candidates need to understand that "Correct Answer Only" responses, directly from the graphical calculator, do not provide evidence of relational thinking.

- Candidates should ensure that any incorrect working or non-relevant part-solution should be crossed out, so that the marker does not include this in their analysis of the response.
- Candidates should ensure that a sufficient degree of accuracy is used throughout a solution when numerical values are being introduced. The rounding of the final answer should only appear at the final stage of the solution.
- Candidates need to be confident in using their calculator efficiently, effectively, and correctly, e.g., Evaluating $\operatorname{cosec}^2\left(\frac{\pi}{4}\right)$, substituting values into exponential functions.

Report on individual achievement standard(s)

Achievement standard 91577: Apply the algebra of complex numbers in solving problems

Assessment

This examination provided candidates with multiple opportunities to display their understanding of the material outlined in the standard; namely:

- quadratic and cubic equations with complex roots
- Argand diagrams
- polar and rectangular forms
- manipulation of surds
- manipulation of complex numbers
- loci
- de Moivre's theorem
- equations of the form $z^n = r\operatorname{cis}\theta$, or $z^n = a + b\mathrm{i}$ where a, b are real and n is a positive integer.

A study of previous years' examination papers and schedules will show that there are many skills that are regularly assessed. Problems related to the many and varied aspects of complex algebra need to be thoroughly understood and learned for the higher levels of success in this achievement standard. In particular, questions requiring the application of de Moivre's theorem occur regularly, so candidates should be thoroughly prepared for this type of problem.

The requirement to show an answer in terms of a specific variable or in in a specific form is not understood or is ignored by some candidates. If the question stipulates that the solution needs to be given in a particular format, then this guidance should be followed, e.g. giving a complex number in the rectangular form or polar form.

Many candidates lacked the necessary knowledge and understanding to form the expression to solve the problem that resulted, or understand the algebra required to make progress toward an answer.

Candidates should realise that an explanation, conclusion, or interpretation at the end of a calculation is a necessary and important part of the solution, and should be included in a solution.

This standard does not cover as much content as the other two external papers. As a result, the paper is perhaps more predictable than the other two achievement standards, and an organised revision programme which paid particular attention to previous papers would pay dividends for the success level of the candidate.

Commentary

There are fundamental skills that candidates who target success in this standard must have the confidence in recognising and using.

The ability to multiply and divide complex numbers in both rectangular and polar form is the most fundamental aspect of this standard. The need for multiplying by a conjugate fraction was required several times in this examination. It should be a skill that is thoroughly rehearsed by candidates in its various forms.

The skill of understanding and finding the modulus and argument of a complex expression was also required several times in the paper.

Candidates should be very familiar with the important definitions and related terminology that need to be recognised and utilised, e.g. magnitude, modulus, argument, conjugate, Argand diagram.

Application of de Moivre’s theorem is another cornerstone technique. Candidates need to be more careful with their answers for questions requiring the use of de Moivre’s theorem when finding solutions to an equation. Candidates need to ensure they calculate the correct argument when converting the original equation to polar form. If this initial angle is incorrect then consequently so is all the following work.

Candidates had trouble representing the modulus correctly, so all other work done in the question was wasted. Using a sketch Argand diagram is recommended to illustrate in which quadrant the point is lying, without relying on a graphical calculator solution.

Grade awarding

Candidates who were awarded **Achievement** commonly:

- expanded and simplified perfect square brackets involving surds, with unknown constants
- evaluated the quotient of two complex numbers in Polar form
- showed an understanding of the argument and modulus of a complex number
- evaluated the modulus of a complex number, with an unknown constant
- recognised how to simplify a complex number by multiplying by the complex conjugate
- manipulated a quadratic equation to identify no real roots, using the discriminant, and solving the resulting inequality
- identified real and imaginary terms in an expression and could group them correctly
- utilised the remainder theorem to find an unknown constant
- attempted a variety of questions
- manipulated complex numbers in either polar or rectangular form
- could find the cartesian equation of a locus.

Candidates who were awarded **Achievement with Merit** commonly:

- solved an equation involving surds and an unknown constant
- demonstrated understanding of, and could interpret, the modulus and argument, and manipulated an equation involving complex numbers to solve it
- showed that the modulus of a complex number must have a positive value
- accurately found the modulus and argument of a complex number which does not lie in the first quadrant
- solved equations involving surds or imaginary numbers
- used de Moivre’s theorem and applied it correctly to find all of the solutions to a given problem
- demonstrated understanding of the meaning of “purely real” or “purely imaginary” complex numbers and could form the equations that resulted
- substituted $z = x + \mathrm{i}y$ into equations, manipulated to separate them into real and imaginary parts
- recognised the using the definitions and terminology of complex numbers
- used appropriate algebraic methods to find complex roots of a cubic polynomial, when given one imaginary root
- formed the cartesian equation of a locus described by a complex equation.

Candidates who were awarded **Achievement with Excellence** commonly:

- used algebra skills to accurately solve an equation, involving quotients of complex numbers, without unnecessary or confusing statements in their working
- recognised when the discriminant should be utilised and when the quadratic formula should be utilised
- demonstrated problem solving skills required to group real and imaginary terms and could

apply the correct algebra to them

- demonstrated understanding of what the modulus symbol required and how it could be applied in a formal proof
- explored problems that not all solutions will be valid and solving accordingly
- completed the required proof by making connections between real and imaginary parts and completing the square
- communicated their thinking clearly and accurately about what they were doing, while completing multi-step problems
- provided clear, logical, and easy-to-follow working, reducing the chance of numerical or algebraic errors
- showed careful thought in recognising when a chosen strategy was not likely to be successful and then reassessing the chosen method
- displayed abstract thinking in solving problems that were not familiar, making links to the information provided in the question
- showed sufficiently strong skills and knowledge on the merit questions as well as displaying strengths on the excellence questions.

Candidates who were awarded **Not Achieved** commonly:

- did not demonstrate basic algebra skills needed to simplify, solve, expand, factorise
- displayed little understanding of the process for completing the square, discriminant or quadratic formula
- did not rationalise a denominator correctly
- did not find the modulus or argument of a complex number
- did not manipulate complex numbers in either Polar or Rectangular form
- converted a complex number from one form to another (rectangular to polar or vice versa) before performing calculations with complex numbers
- did not demonstrate how to convert to polar form, without the use of a calculator
- generally relied too heavily on the use of a graphics calculator
- showed no understanding of interpreting or using de Moivre's Theorems
- showed no understanding of, or how to interpret complex numbers represented on an Argand diagram
- showed no understanding of or how to apply correctly the Remainder Theorem
- made too many basic errors in simplifying and applying algebraic methods
- did not multiply surds together correctly
- did not maintain consistency when solving inequations or equations
- did not attempt a sufficiently large proportion of the assessment.

Achievement standard 91578: Apply differentiation methods in solving problems

Assessment

This examination provided candidates with multiple opportunities to display their understanding of the material outlined in the standard; namely:

- derivatives of power, exponential, and logarithmic (base e only) functions
- derivatives of trigonometric (including the reciprocal trigonometric) functions
- optimisation
- equations of tangents and normals
- maxima, minima and points of inflection
- related rates of change
- derivatives of parametric functions

- chain, product, and quotient rules
- properties of graphs (limits, differentiability, continuity, concavity).

Candidates must ensure that they are fully confident in their own ability to be able to differentiate all various forms of the functions, including when the use of the chain, product, and quotient rules are necessary. Many of the candidates displayed the necessary understanding to be able to interpret the application using differentiation methods, but were let down by errors in their methods of differentiation. Success can only be awarded if the applications of the differentiation rules are displayed accurately and error-free.

In order to solve the application of differentiation problems, candidates need to have sound algebra skills, as well as strong knowledge of the various differentiation methods, which come from continual practice of a wide variety of possibilities. Many candidates are not sufficiently confident enough to solve quadratic equations or equations involving the use of exponentials and logarithms. Candidates should not be relying solely on their graphical calculators to solve such equations as the inclusion of unknown constants will generally eliminate this route. For all levels of success, both necessary skills of differentiation and algebra are necessary.

Modelling for optimisation problems continues to be a skill that only the best candidates can handle. Candidates need to work with their teachers to find ways to gain greater confidence in these types of problems. The optimisation problem, involving a helicopter chasing a speeding car, (Question Two (e)) was too hard for all but the very top candidates. Candidates need a lot more practice at understanding how to deal with creating appropriate models within an optimisation problem.

Many candidates were confused when the problem connected their differentiation and co-ordinate geometry knowledge. (Question One (c)). These types of problems should be included in all teaching programmes. It was evident from the errors seen that candidates were not sufficiently confident and experienced in this type of problem solving, whilst relying too heavily on the use of the graphical calculator.

Commentary

Candidates are reminded that evidence of clear working and methods must be shown at all levels of the examination. Answers provided without appropriate supporting working is unlikely to be rewarded fully. Candidates should follow the guidance in some questions that stipulate “you do not need to simplify your answer”.

Strong algebraic skills, as well as confidence in all of the differentiation methods, are crucial for success in this standard: expanding, factorising, simplifying expressions, manipulating algebraic fractions, and solving various equations – especially quadratic equations, but also equations involving exponential, logarithmic, rational expressions, and trigonometric functions. Similarly, candidates should be aware that the square root of a value will always lead to two possible solutions, one of which may not be valid in a particular question part.

The correct use of brackets and accurate mathematical statements remains a challenge for many Calculus candidates. The ability to avoid making errors, or having an effective self-checking system to eliminate any mistakes, is important.

Many candidates were unable to differentiate correctly to find the first and second derivatives of the function in Question Three (e), as errors were caused by the constant and then they failed to recognise that the subsequent differentiations did not require the use of the product rule.

Candidates frequently omitted a factor of the differentiation, which led to an incorrect differentiation, so no credit could be given for this question part, even for candidates who subsequently managed to continue with their solution.

Candidates need to be aware and knowledgeable regarding completing a proof in an appropriately formal mathematical manner. Many students were able to differentiate successfully but were not able to complete the proof in a sufficiently formal manner (Question One (d), Question One (e), Question Three (e)).

Grade awarding

Candidates who were awarded **Achievement** commonly:

- used the chain rule, product rule, and quotient rule correctly in combination with power functions, trigonometric functions, parametric functions, exponential and logarithmic functions

- solved quadratic, exponential, and power equations resulting from differentiating a function
- used differentiation to find the gradient of tangents
- found the rate of change of a function, at a given value
- found an appropriate model to solve an optimisation problem and then correctly differentiated their model
- found the two components of a related rates of change problem and then correctly differentiated them
- calculated the x-coordinate correctly for any stationary points and inflection points
- recognised features of gradient, differentiability, and limit from a piecewise graph
- discarded $x = 0$ as a solution, where appropriate
- used differentiation to find the gradient of a given function
- showed sufficient algebraic skills to simplify their answer having differentiated a function, when required to do so
- used their calculators confidently and knowledgeably to evaluate differentiated functions, including the correct use for trigonometric and exponential functions.

Candidates who were awarded **Achievement with Merit** commonly:

- solved equations involving logarithmic and exponential functions correctly to find the coordinates of stationary points and inflection points, demonstrating confident algebraic skills
- used confidently and correctly the product quotient rule, and chain rule when differentiating functions
- communicated their solutions distinctly, with clear working, in order to demonstrate their understanding and method, ensuring that the requirements of the question were fulfilled
- solved a related rates problem
- used relational thinking to connect co-ordinate geometry and differentiation knowledge and apply to gradients of points, tangents, and normals
- found the equation of a tangent on a parametric function graph, involving unknown constants
- demonstrated reliable and accurate algebra skills when solving problem, including exponential and logarithmic functions
- demonstrated the procedure to set up models for optimisation questions, in terms of one variable, before correctly differentiating it and then solving the resulting equations
- found second differentials of an exponential function
- manipulated surds and aspects of trigonometry correctly, resulting from differentiating
- located and identified the nature of stationary points and inflection points, utilising a variety of options of methods.

Candidates who were awarded **Achievement with Excellence** commonly:

- demonstrated high quality algebraic skills when solving problems resulting from differentiating, especially when simplifying expressions
- set-up appropriate models and then interpret them correctly to solve the problem using differentiation
- confidently found and used the second derivative of functions when solving problems
- demonstrated abstract thinking whilst applying differentiation methods in solving unfamiliar problems, utilising precise and accurate algebraic methods
- completed a proof using logical, clear working and communicating each step of a ‘proof’ question with sufficient detail
- formed and interpret a related rates of change problem to find the speed of the car being chased by a police helicopter.

Candidates who were awarded **Not Achieved** commonly:

- showed a lack of confidence and accuracy in applying the chain rule, product rule, or quotient rule in combination with power functions, trigonometric functions, parametric functions, exponential and logarithmic functions

- appeared confused on how, when, and which differentiation method should be applied
- did not recognise when it was necessary to apply the product and quotient rules for differentiation
- made careless errors in a solution, reflecting a lack of an effective self-checking process
- did not solve a resulting equation algebraically, even though they had found the correct derivative
- did not correctly factorise and solve quadratic equations, even though they had found the correct derivative
- demonstrated poor algebraic skills such as inaccurate expanding, factorising, cancelling of factors, manipulating algebraic fractions, knowledge of surds, careless errors, and providing only one solution for the square root of a value
- did not recognise features of gradient, differentiability, and limits from a graph
- did not actually answer the requirements of the question
- made errors in the use of their calculator, especially when evaluating terms involving trigonometric functions and exponential functions
- omitted the necessary brackets in their solution.

Achievement standard 91579: Apply integration methods in solving problems

Assessment

This examination provided candidates with multiple opportunities to display their understanding of the material outlined in the standard; namely:

- integrating power, polynomial, exponential (base e only), trigonometric, and rational functions
- reverse chain rule, trigonometric formulae
- rates of change problems
- areas under or between graphs of functions, by integration
- finding areas using numerical methods, e.g. the rectangle or trapezium rule
- differential equations of the forms $y' = f(x)$ or $y'' = f(x)$ for the above functions or situations where the variables are separable (e.g. $y' = ky$) in applications such as growth and decay, inflation, Newton's Law of Cooling, and similar situations.

While the ability to integrate the types of functions listed in the bullet points above is critical to success, so is having the algebra skills required to solve the related problems. These algebra skills are necessary for all three external standards in the Calculus course and their constant practice and reinforcement, including the essential use of brackets to communicate methods and maintain accuracy of solutions, should be a focus for all candidates.

Candidates must ensure that they are fully confident in their own ability to be able to integrate all various forms of the functions, illustrating their strengths in recognising which method is applicable. Many of the candidates displayed the necessary understanding to be able to interpret the application using appropriate integration methods but were let down by errors in their methods of integration. Success can only be awarded if the methods are applied accurately and free of any errors.

Commentary

Candidates should be strongly discouraged from only attempting the "Excellence" part of each question. It was very common for candidates who did this to make an error on one or more of these, and ultimately under-perform on their intended target. Many candidates enhanced their overall final grade by attempting all parts of all questions, thereby gaining grade u or grade r rewards on the merit or excellence question parts.

Candidates should be encouraged to use the formal and correct terminology throughout the solution to a problem, in order to avoid careless errors, and to guide the candidate through each step of the necessary working and calculations, including the accurate use of the constant of integration. Within the teaching and learning programme, it needs to be stressed that candidates must correctly use the constant of integration. The constant of integration should not be omitted and, additionally, its value need not necessarily be zero.

Candidates are reminded that evidence of the correct use of integration methods must be clearly demonstrated, answers from a graphical calculator are unlikely to attain higher than an “achieved” grade, at best.

Candidates frequently omitted a factor of the integration which led to an incorrect integration, so no credit could be given for this question part, even for candidates who managed to arrive at the correct solution.

It is important that candidates read the question fully and carefully. Many candidates misread Question two (e) as “square root of the volume” instead of “square of the volume”, which adversely affected their overall grade.

Grade awarding

Candidates who were awarded **Achievement** commonly:

- integrated basic functions correctly, including polynomial functions, rational functions, exponential functions, and trigonometric functions
- successfully wrote surd expressions into exponent form
- manipulated functions and expressions into a form which could be integrated
- attempted the majority of questions, in particular integrating any functions in the paper that needed to be integrated
- were familiar with the fundamental theorem of integration
- used Simpson’s Rule correctly
- solved simple differential equations
- solved simple kinematics problems, needing the use of integration methods
- provided and evaluated the constant of integration, given the necessary information.

Candidates who were awarded **Achievement with Merit** commonly:

- integrated a rational function by using either long division or substitution
- used trigonometric identities correctly in order to be able to apply an appropriate integration method
- solved differential equations by successfully separating variables, and also including calculating the arbitrary constant
- recognised when to use the natural logarithm to integrate
- evaluated an area that lies between two functions
- applied integration methods in relation to kinematics
- recognised which integration method could be applied to a specific problem
- were well-rehearsed with regards to certain popular integrations requiring relational thinking
- calculated the finite area between two curves and / or were able to calculate the area between a curve and the *x*-axis, recognising that areas below the *x*-axis would be “negative areas”.

Candidates who were awarded **Achievement with Excellence** commonly:

- carried out appropriate algebraic manipulations to attain an expression suitable to integrate
- recognised the need to use the difference of two squares identity to simplify the algebra after separating the variable in a differential equation
- demonstrated the knowledge and understanding in order to solve problems involving an area needing to be subdivided into suitable separate areas
- applied understanding in recognising how to separate the variables in a differential equation and form expressions to enable integration methods to be applied, and also to work with exact values, manipulating logarithms and avoid errors with negative values
- demonstrated understanding to use correct and appropriate mathematical statements
- demonstrated the ability to form a differential equation for a contextual problem
- correctly separated the variables and then used the appropriate information to find the value

of the constants in terms of p, and hence solve the problem
- had sufficiently strong skills and knowledge on the merit questions as well as displaying strengths on the excellence questions.

Candidates who were awarded **Not Achieved** commonly:

- demonstrated insufficient understanding to recognise the appropriate method and consequent accurate integration when integrating power, polynomial, exponential, trig, and rational functions
- believed that all rational expressions led to a integration involving $\ln(f(x))$
- did not algebraically rearrange, simplify, or manipulate a function so that it was in a suitable form, which could then be integrated
- lacked accuracy and appropriate usage of Simpson's Rule to complete a numerical method, in particular, not calculating the correct necessary interval
- did not recognise how integration could be used to evaluate an area
- were not sufficiently confident with the rules of indices, so that errors appeared when manipulating $\sqrt{x}$ into an alternative appropriate format involving exponents
- confused differentiation and integration methods
- did not separate the variables in a differential equation
- omitted a factor of the integration, which consequently then led to an incorrect integration
- relied too heavily on the use of their graphical calculator when evaluating definite integrals, and omitted the evidence of the integration method used.
