2

SUPERVISOR'S USE ONLY

91262

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NEW ZEALAND QUALIFICATIONS AUTHORITY
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QUALIFY FOR THE FUTURE WORLD
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Level 2 Mathematics and Statistics, 2017

91262 Apply calculus methods in solving problems

2.00 p.m. Friday 24 November 2017
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply calculus methods in solving problems.	Apply calculus methods, using relational thinking, in solving problems.	Apply calculus methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Make sure that you have Formulae Sheet L2–MATHF.

Show ALL working.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

You must show the use of calculus in answering all questions in this paper.

Check that this booklet has pages 2–12 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 23

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QUESTION ONE

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(a) A function f is given by $f(x) = x^5 + 3x^2 - 7x + 2$.

Find the gradient of the graph of the function at the point where $x = 1$.

$f'(x) = 5x^4 + 6x - 7$

$f'(1) = 5(1^4) + 6(1) - 7$

$= 5 + 6 - 7$

$\therefore m = 4$ //

(b) Find the equation of the tangent to the graph of the function

$f(x) = 6 + 14x - 2x^3$

at the point (2,18) on the graph.

$f'(x) = 14 - 6x^2$

$f'(2) = 14 - 6(2^2)$

$= -10$

$y - y_1 = m(x - x_1)$

$y - 18 = -10(x - 2)$

$y - 18 = -10x + 20$

$y = -10x + 38$ //

(c) The movement of an object is recorded from the time it passes a fixed point. After t seconds it has a speed v m s^{-1}, which can be modelled by the function

$v(t) = 0.5t^2 - 2t + 1$

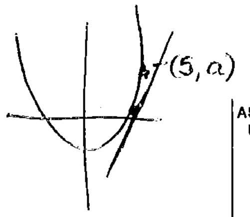
Use calculus to find how long it takes to reach an acceleration of 2.8 m s^{-2}.

$a(t) = t - 2$

$2.8 = t - 2$

$t = 4.8$ seconds //

$y-y_1=m(x-x_1)$



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(d) A tangent to the graph of the function $f(x) = 3x^2 - 4x$ has a gradient of 2, and passes through the point $(5,a)$, where a is a constant.

Find the value of a.

$f'(x) = 6x - 4$

$\cancel{2 = 6(5) - 4}$

$\cancel{= 26}$

$\cancel{y-y_1 = m(x-x_1)}$

$\cancel{y-a = 2(x-5)}$

$\cancel{y-a = 2x-10}$

$\cancel{y = 2x-10+a}$

$f'(x)=6x-4$

$2=6x-4$

$6=6x$

$x=1$

$f(1)=3(1)^2-4(1)$

$=-1$

$y-y_1=m(x-x_1)$

$y+1=2(x-1)$

$y+1=2x-2$

$y=2x-3$

$y=2(5)-3$

$y=7$

For coordinate $(5,a)$, $a=7$

$f'(x)=0$

(e) The function $f(x) = x^3 + ax^2 + bx + 2$ has turning points when $x = -1$ and $x = 3$.

Find the values of a and b.

$f'(x) = 3x^2 + 2ax + b$

$\cancel{(x+1)(x-3)} \rightarrow$ ~~turning points~~

$\cancel{x^2-2x-3} \rightarrow$ ~~x by 3 to have $a=3$~~

$\cancel{3x^2}$

$0 = 3(-1)^2 + 2a(-1) + b$

$0 = 3 - 2a + b$

$-3 = -2a + b$

① $-3 = b - 2a$

$0 = 3(3^2) + 2a(3) + b$

$0 = 27 + 6a + b$

$-27 = 6a + b$ ②

Eliminate ① − ②

$-3 = -2a + b$ −

$(-27 = 6a + b)$

$24 = -8a$

$a = -3$

sub $a=-3$ into ①

$-3 = b - 2(-3)$

$-3 = b + 6$

$b = -9$ //

$\therefore f(x) = x^3 - 3x^2 - 9x + 2$

E8

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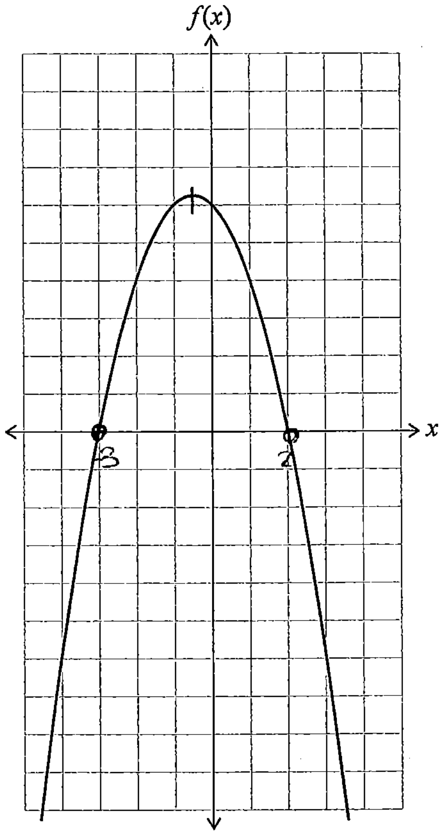
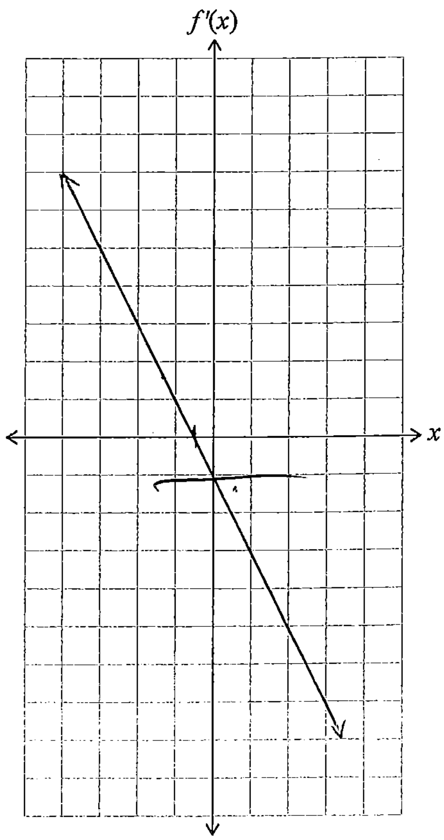
QUESTION TWO

(a) The diagram below shows the graph of the function $y = f(x)$



Sketch the graph of the gradient function $y = f'(x)$ on the axes below.

Both sets of axes have the same scale.



If you need to redraw this graph, use the grid on page 11.

(b) The graph of a function $f(x) = 2x^3 + bx^2 - 2$ has a turning point when $x = -1$.

Find the value of b.

$f'(x) = 6x^2 + 2bx$

$0 = 6(-1)^2 + 2b(-1)$

$0 = 6 - 2b$

$2b = 6$

$b = 3$

(c) Use calculus to show that the line $y = 15x - 12$ is a tangent to the graph of the function $f(x) = 4x^2 - x + 4$.

~~$f'(x) = 8x - 1$~~

$f'(x) = 8x - 1$

$15 = 8x - 1$

$16 = 8x$

$x = 2$

$\therefore$ Tangent at $x = 2$

$f(2) = 4(2)^2 - 2 + 4$

$= 16 - 2 + 4$

$= 18$

$\therefore$ y-ordinate is 18

$y - y_1 = m(x - x_1)$

Sub $x = 2$, ~~and~~ $y = 18$ and $m = 15$

$y - 18 = 15(x - 2)$

$y - 18 = 15x - 30$

$y = 15x - 12$

$\therefore$ $y = 15x - 12$ is a tangent to the graph of the function $f(x) = 4x^2 - x + 4$ at the point $(2, 18)$

(d) Use calculus to find the value of k if the line $y = 6x + k$ is a tangent to the graph of the function $f(x) = x^2 + 2x - 1$.

$f'(x) = 2x + 2$

$6 = 2x + 2$

$4 = 2x$

$x = 2$

$f(2) = 2^2 + \overset{(2 \times 2)}{~~2x~~} - 1$

$= 7$

There is more space for your answer on the following page.

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$y - y_1 = m(x - x_1)$ $\quad\therefore k = -5$

$y - 7 = 6(x - 2)$

$y - 7 = 6x - 12$

$y = 6x - 5$

(e) Use calculus to prove that the graph of the function

$y = x^3(3 - x)$

has a local maximum when $x = \frac{9}{4}$

Justify that the turning point is a local maximum.

~~$\frac{dy}{dx} =$~~

$y = 3x^3 - x^4$ $\qquad y = 3(\frac{9}{4})^3 - (\frac{9}{4})^4$

$\frac{dy}{dx} = 9x^2 - 4x^3$ $\qquad = \frac{2187}{256}$

$0 = 9x^2 - 4x^3$

~~2.25, 0, 0~~

$x = \frac{9}{4}, 0$

Region	$x < 0$	$x = 0$	$0 < x < \frac{9}{4}$	$x = \frac{9}{4}$	$x > \frac{9}{4}$	
Test	$x = -1$	$x = 0$	$x = 1$	$x = \frac{9}{4}$	$x = 3$	
$f'(x)$	13	0	5	0	−27	
Gradient	/	—	/	—	\	

$\therefore$ The graph of the function $y = x^3(3 - x)$ has a local maximum at $(\frac{9}{4}, \frac{2187}{256})$. It is a local maximum within the domain $x > 0$ because when $0 < x < \frac{9}{4}$ the gradient is increasing and when $x > \frac{9}{4}$ the gradient is decreasing.

E8

ASSESSOR'S USE ONLY

QUESTION THREE

(a) The gradient graph of a function $f(x)$ is given by

$$f'(x) = 6x^2 - 2x + 4$$

The point (1,3) lies on the graph.

Find the equation of the function $f(x)$.

$f(x) = \frac{6}{2+1}x^{2+1} - \frac{2}{1+1}x^{1+1} + 4x + c$

$= 2x^3 - x^2 + 4x + c$

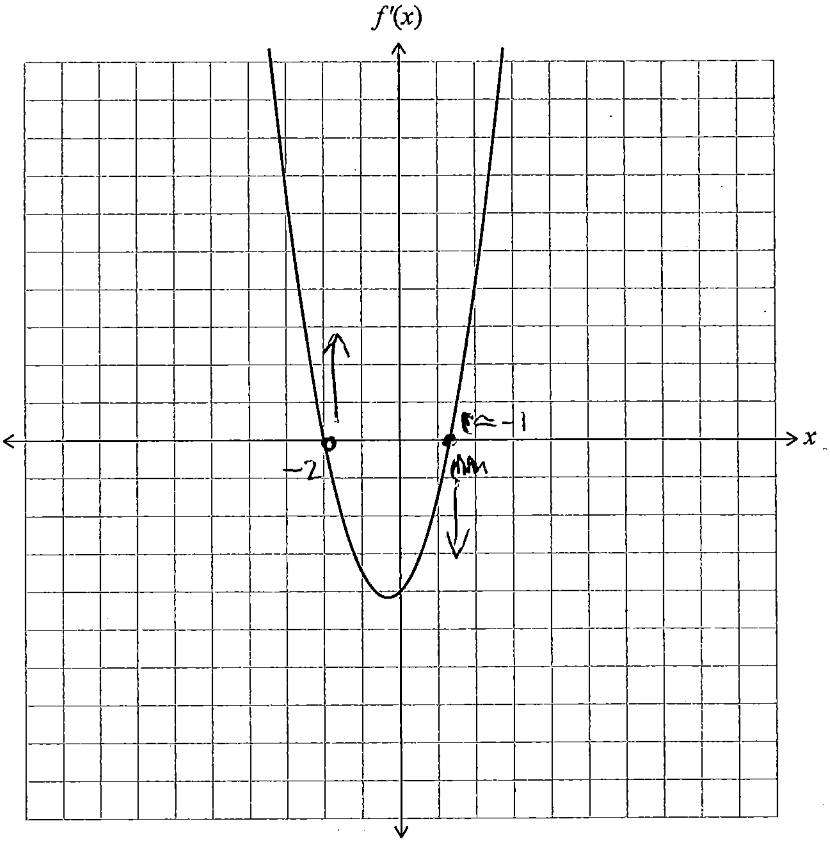
$3 = 2(1^3) - (1^2) + 4(1) + c$

$3 = 5 + c$

$c = -2$

$\therefore f(x) = \underline{2x^3 - x^2 + 4x - 2}$ //

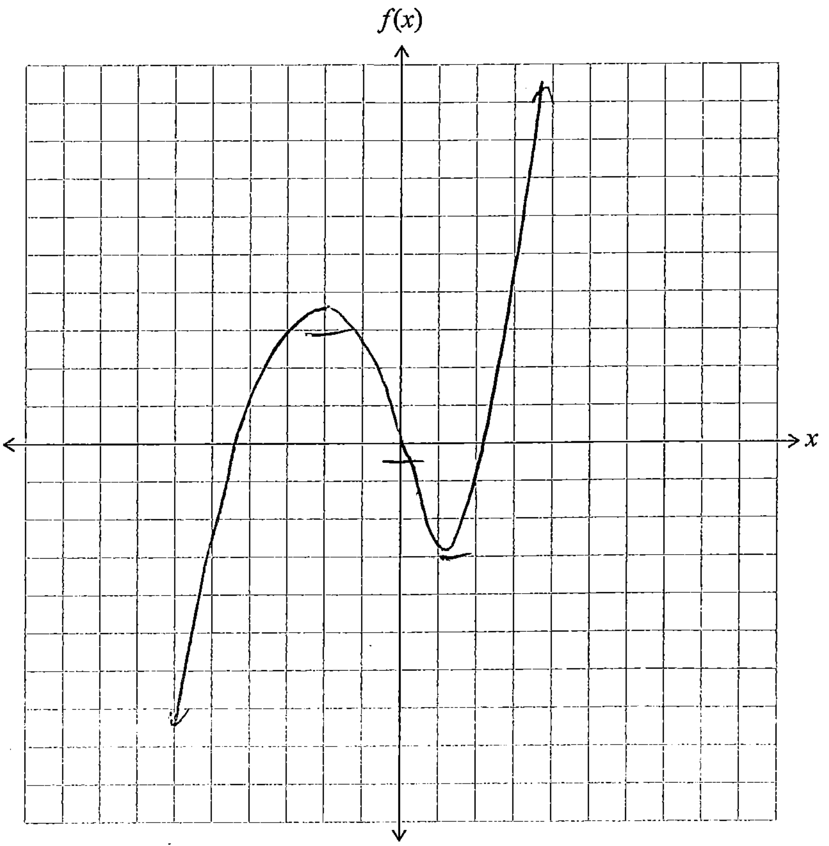
(b) The diagram below shows the graph of a gradient function $y = f'(x)$.



The point (0,0) is on the graph of the function $y = f(x)$.

On the axes below sketch the function $f(x)$.

Both sets of axes have the same scale.



If you need to redraw this graph, use the grid on page 11.

ASSESSOR'S USE ONLY

(c) An object can move in either direction on a straight track and has a constant acceleration of -4 cm s^{-2}.

A fixed point P is marked on the track.

When a recording of the object's motion begins, the object:

- is 12 cm from P, $c = 12$
- is moving away from P, and
- has a velocity of 6 cm s^{-1}. $c = 6$

(i) Using calculus, find the speed of the object 5 seconds after its motion began being recorded.

$a(t) = -4$

$v(t) = -4t + c$

when $t = 0$, $v = 6$

$\therefore v(t) = -4t + 6$

$v(5) = -4(5) + 6$

$= -14 \text{ m/s}$

$v(t) = 0$

(ii) What is the maximum distance of the object from the point P?

Justify that this is the maximum distance.

$s(t) = \frac{-4}{1+1} t^{1+1} + 6t + c$

$= -2t^2 + 6t + c$

when $t = 0$, $s = 12$

$\therefore s(t) = -2t^2 + 6t + 12$

$v(t) = -4t + 6$

$0 = -4t + 6$

$4t = 6$

$t = 1.5$ seconds

$s(1.5) = -2(1.5^2) + 6(1.5) + 12$

$= -2(2.25) + 6(1.5) + 12$

$= 16.5$ m

$\therefore$ The maximum distance of the object from point P is 16.5m. it is the maximum distance because when its derivative was equated to 0, it was determined that the max. distance was reached at 1.5 seconds, which, when substituted into the s(t) equation, gave 16.5m as the answer.

Question Three continues on the following page.

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(d) Find the maximum volume of an open box (i.e. a box with a base and sides, but no lid) that can be made from a rectangular piece of cardboard measuring 20 cm by 30 cm, by removing the corner squares and folding along the dotted lines.

Justify that this is the maximum volume.

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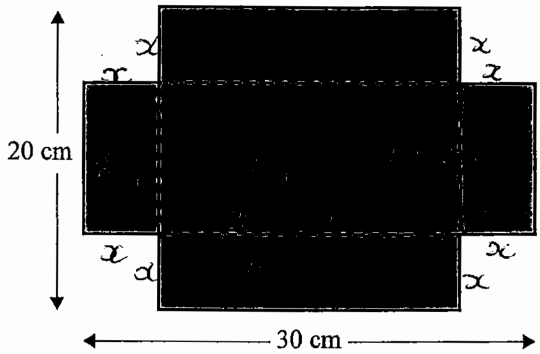

raw √

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ubstitute

erive

quate to 0

ack to original

st

?

~~V=lwh~~ $V = lwh$ ← sub into this one!

$A = 2A_3 + 2A_2 + A_1$

$= 2(20-2x \times x) + 2(30-2x \times x) + (30-2x)(20-2x)$

$= 2(20x-2x^2) + 2(30x-2x^2) + (600-60x-40x+4x^2)$

$= 40x - 4x^2 + 60x - 4x^2 + 600 - 100x + 4x^2$

$= -4x^2 + 60$?

$V = (30-2x)(20-2x)x$

$= (600 - 100x + 4x^2)x$

$= 600x - 100x^2 + 4x^3$

$V' = 600 - 200x + 12x^2$

$0 = 600 - 200x + 12x^2$

~~$0 = 50 - \frac{50}{3}x + x^2$~~

~~$-50 = \frac{50}{3}x + x^2$~~

~~$-150 = -50x + 3x^2$~~

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-200) \pm \sqrt{(-200)^2 - 4(600)(12)}}{2 \times 12}$

$= \frac{200 \pm \sqrt{40000 - 28800}}{24}$

$= \frac{200 \pm \sqrt{11200}}{24}$

$x_2 = 12.74291885,$

$x_1 = 3.923747815$

Reg	$x < x_1$	$x = x_1$	$x_1 < x < x_2$	$x = x_2$	$x > x_2$
Test	$x=0$	$x=x_1$	$x=5$	$x=x_2$	$x=14$
$f'(x)$	600	0	-100	0	182
Grad.	/	—	\	—	/

$V = 600x - 100x^2 + 4x^3$

$V(3.92) = 600(3.92) - 100(3.92)^2 + 4(3.92)^3$

$= 1056.305895$

$= 1056.31 \text{ cm}^3$

∴ The maximum volume of the open box is 1056.31cm^3.

It is the maximum volume because it has the x-ordinate value of the maximum point on it's cubic curve graph shape as the x-value / dimensions.

E7

Subject:	Mathematics	Standard:	91262	Total score:	23

Q	Grade score	Annotation
1	E8	1(a) Correct derivative and gradient. 1(b) Correct gradient and equation for tangent. 1(c) Correct a(t) equation and substitution to find t at a = 2.8. 1(d) Correct derivative and solving to get x = 1, y = -1. Correct tangent equation and solution at x = 5 to obtain a. 1(e) Correct derivative and f'(x) = 0. Correct substitution of x = -1 and x = 3 with simultaneous equations correctly formed then solved (could have used GC to solve equations).
2	E8	2(a) Correct slope and x intercept. 2(b) Correct derivative and f'(x) = 0 solved to find b. 2(c) Correct f'(x) equated to slope of y = 15x -12 to find x and y. Equation of tangent passing through this point found. Evidence that (2,18) lies on f(x). 2(d) f'(x) = 6 and intersecting point of tangent found. Tangent equation formed and used to find k. 2(e) Correct derivative and solutions to y' = 0 for turning points. Investigated slope either side of turning points to justify the maximum.
3	E7	3(a) Correct anti-differentiation of f'(x) and f(x) obtained for (1,3). 3(b) Correct shape, (0,0) intersect with max and min points at x intercepts. 3(c)(i) Correct equation for v and speed at t=5s found. 3(c)(ii) Correct t= 1.5s found with correct s(t) and distance of 16.5cm. Insufficient justification that this is a maximum distance. 3(d) Correct equation for volume and derivative, correct solutions for max/min situation. A max determine by investigating the rate of change either side of turning points, the maximum volume clearly stated.