91262

912620

2

SUPERVISOR'S USE ONLY

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 2 Mathematics and Statistics, 2018

91262 Apply calculus methods in solving problems

9.30 a.m. Wednesday 14 November 2018
Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply calculus methods in solving problems.	Apply calculus methods, using relational thinking, in solving problems.	Apply calculus methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Make sure that you have Formulae Sheet L2–MATHF.

Show ALL working.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

You must show the use of calculus in answering all questions in this paper.

Check that this booklet has pages 2–15 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 21

ASSESSOR'S USE ONLY

QUESTION ONE

ASSESSOR'S USE ONLY

(a) A function f is given by $f(x) = x^3 - 6x + 2$.

Find the gradient of the graph of the function at the point where $x = 4$.

$f'(x) = 3x^2 - 6$

$m = 3(4)^2 - 6$

$m = (3 \times 16) - 6$

$m = 48 - 6$

$= 42$

gradient is $\underline{42}$

u

(b) A rectangle is expanding in area so that at all times its length is three times its width.

Find the rate of change of the area of the rectangle with respect to its width when the area of the rectangle is 75 cm^2.

$wL = 75 = A$

$L = 3w$

$A = wL$

$= w(3w)$

$= 3w^2$

$A' = \underline{6w}$

$3w^2 = 75$

$w^2 = 25$

$w = \pm 5$

$A'(5) = \underline{30}$ cm/s

L NOT NEGATIVE AS AREA //

r

ASSESSOR'S USE ONLY

(c) The derivative of a function f is given by $f'(x) = -3x^2 + 12x$.

The graph of the function has a local minimum at the point (0,5).

Use calculus to find the value of the local maximum of the function.

$-3x^2 + 12x = 0$

$-3x(x+4) = 0$

$x = 0$ or $x = 4$

Min at $x = 0$ (stated above)

So Max at $x = 4$

Proof: $f''(x) = -6x + 12$.

$f''(4) = -12 < 0$ ∴ Local max at $x = 4$

u.

(d) Use calculus to find the values of x for which the graph of the function $f(x) = \frac{2}{3}x^3 + \frac{9}{2}x^2 - 5x - 18$ is increasing.

positive cubic

$f'(x) = 2x^2 + 9x - 5$

$0 = 2x^2 + 9x - 5$

$x = 0.5$ and $x = -5$ (GC)

The graph of the function is increasing when x is less than -5 and when x is more than 0.5, because the graph is a positive cubic

$-5 > x > 0.5$

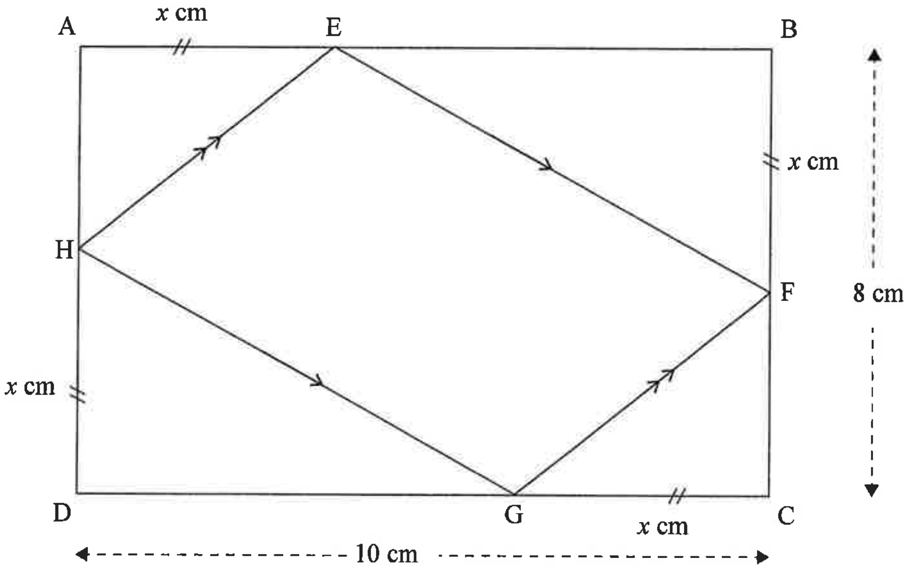
t

ASSESSOR'S USE ONLY

(e) A rectangle ABCD measures 10 cm by 8 cm. A parallelogram EFGH can be drawn inside the rectangle, as shown in the diagram below.

Suppose that the distance from each corner of the rectangle to the next vertex of the parallelogram, in a clockwise direction, is x cm.

That is, AE = BF = CG = DH = x.



Use calculus to find the smallest possible area that the parallelogram can have.

Justify that your answer is a minimum.

$A = 80 - [\frac{1}{2} \times x \times (8-x)] \times 2 - [\frac{1}{2} \times x \times (10-x)] \times 2$

$= 80 - (4x - \frac{1}{2}x^2) \times 2 - (5x - \frac{1}{2}x^2) \times 2$

$= 80 - (8x - x^2) - (10x - x^2)$

$= 80 - 8x + x^2 - 10x + x^2$

$= 80 - 18x + 2x^2$

$= 2x^2 - 18x + 80$

$A' = 4x - 18$

$0 = 4x - 18$

$18 = 4x$

$x = 4.5$

$A = 2(4.5)^2 - 18(4.5) + 80$

$= 39.5$

$\therefore$ Minimum Area is 39.5 cm^2. A

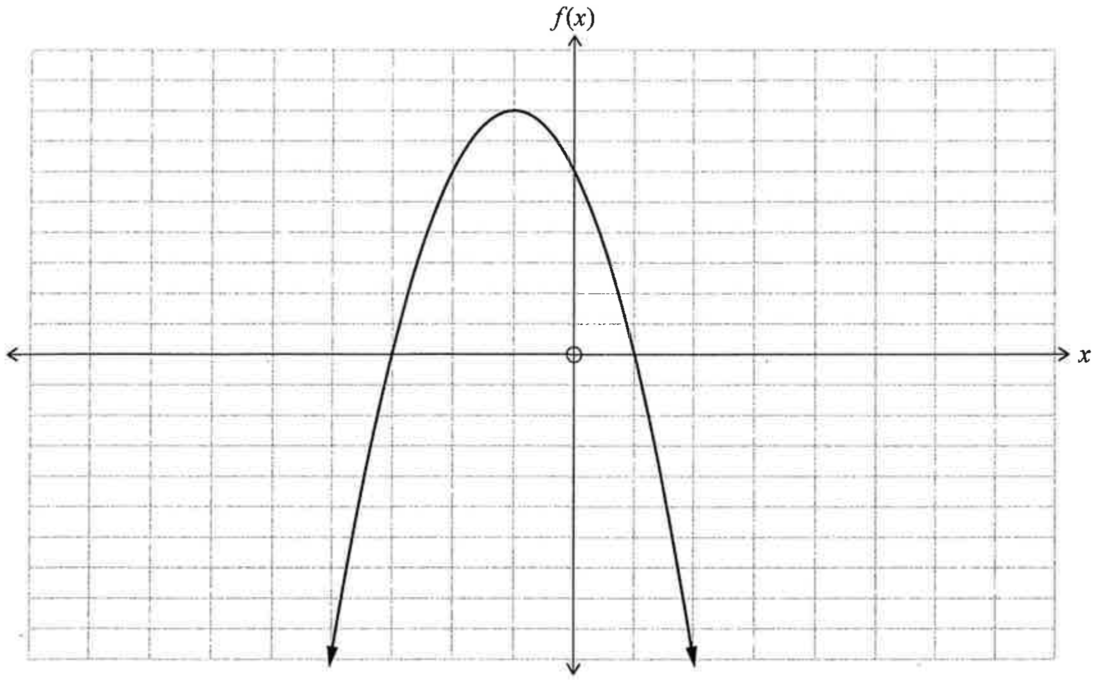
r

E7

QUESTION TWO

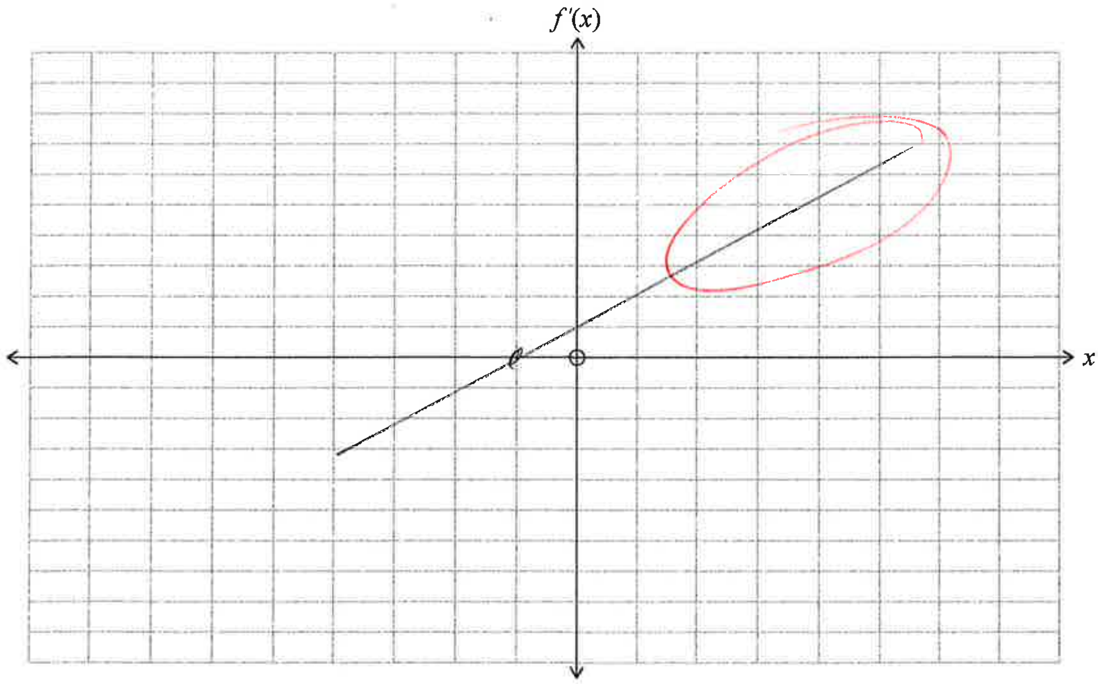
ASSESSOR'S USE ONLY

(a) The graph of a function $y = f(x)$ is shown on the axes below.



Sketch the graph of the gradient function $y = f'(x)$ on the axes below.

Both sets of axes have the same scale.



N

If you need to redo this question part, use the grids on page 12.

(b) A skyrocket is projected into the air so that t seconds after it is launched, its height, h metres, above the ground is given by

$$h(t) = 39.2t - 4.9t^2.$$

What is the maximum height that the skyrocket will reach?

$h'(t) = 39.2 - 4.9t$

$0 = 39.2 - 4.9t$

$t = \frac{-39.2}{-4.9}$

$t = 8$

$h(8) = 39.2 \times 8 - 4.9 \times 8^2$

$= 313.6 - 313.6$

$= 0$?

ASSESSOR'S USE ONLY

n

(c) Adam is operating his drone. It is moving in a straight line and t seconds after passing a tree its acceleration, a m s^{-2}, is given by

$$a(t) = 6 - 12t.$$

Two seconds after the drone passed the tree, its velocity was 20 m s^{-1}.

How far was the drone from the tree when its velocity was 20 m s^{-1}?

$V(t) = 6t - 6t^2 + C.$

$20 = 6(2) - 6(2)^2 + C.$

$20 = 12 - 24 + C.$

$20 = -12 + C$

$C = 32$

So $V(t) = 6t - 2t^2 + 32$

$s(t) = 3t^2 - 2t^3 + 32t + C.$

$0 = 3(0)^2 - 2(0)^3 + 32(0) + C.$

$C = 0$

$s(t) = 3t^2 - 2t^3 + 32t.$

$s(t) = 3(2)^2 - 2(2)^3 + 32(2)$

$= 3 \times 4 - 2 \times 8 + 64$

$= 12 - 16 + 64$

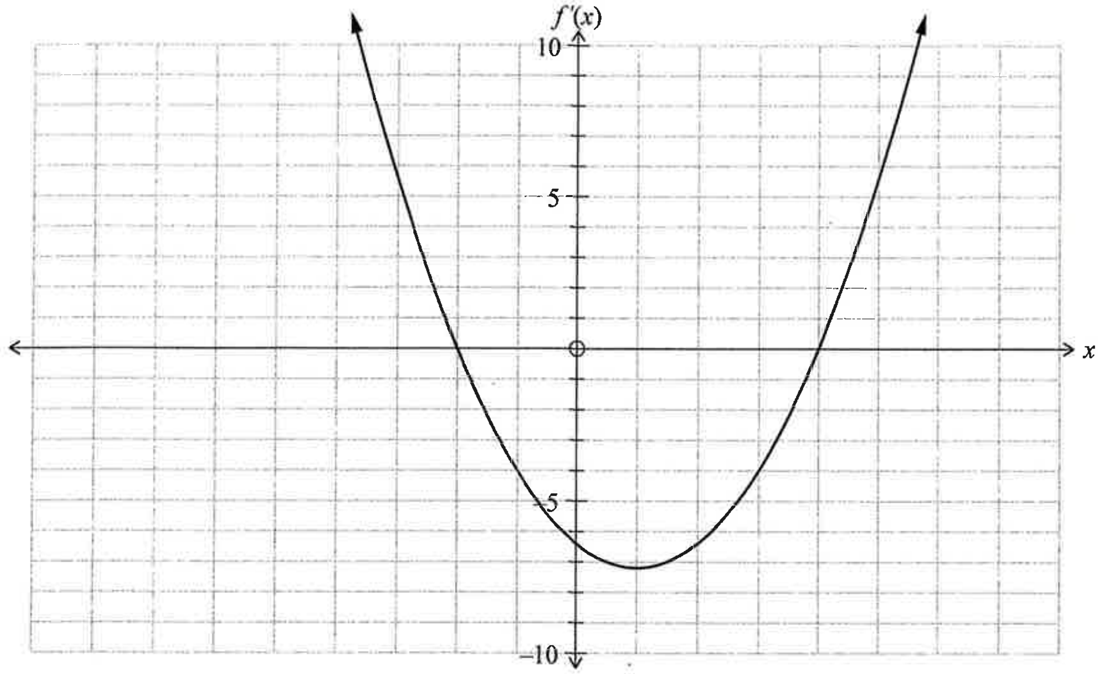
$s(t) = 60$

The drone was 60 metres from the tree when its velocity was 20 m s^{-1}

t

(d) The diagram below shows the graph of the gradient function $y = f'(x)$ of a function $y = f(x)$.

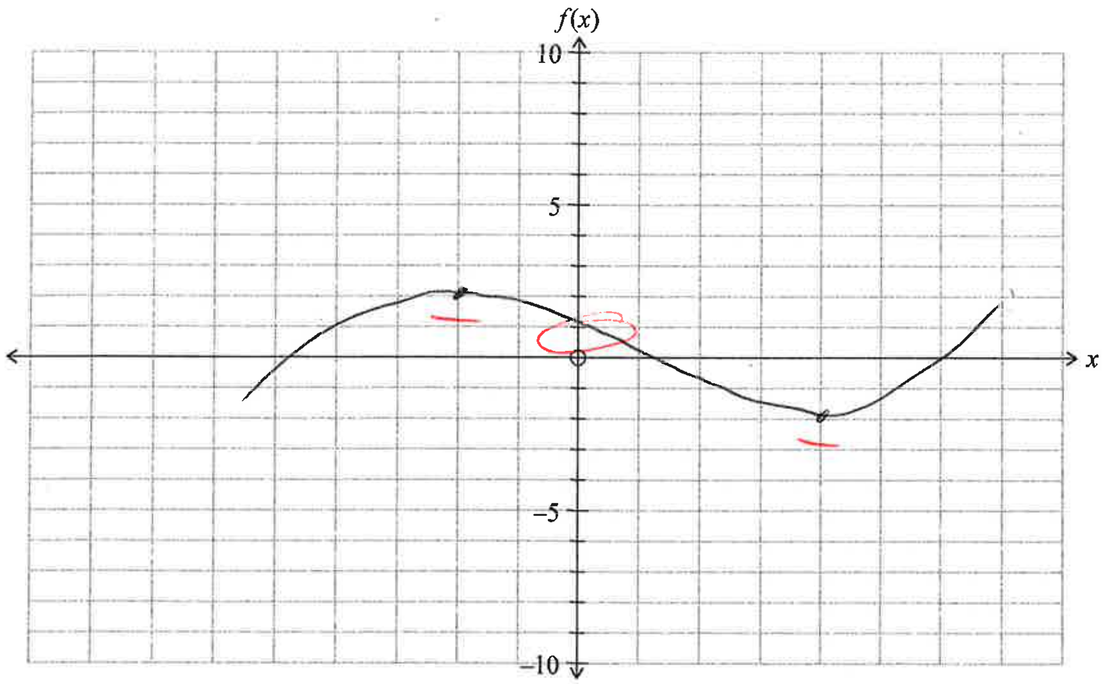
ASSESSOR'S USE ONLY



The graph of the function $y = f(x)$ passes through (0,3).

On the axes below sketch the graph of the function f.

Both sets of axes have the same scale.



4

If you need to redo this question part, use the grids on page 13.

(e) The graph of the function $y = x^3 - 6x^2 + kx - 5$ has a turning point at $x = 3$.

Use calculus methods to find the coordinates of both turning points.

Determine the nature of each turning point, justifying your answer.

$\frac{dy}{dx} = 3x^2 - 12x + k$

$0 = 3x^2 - 12x + k$

$0 = 3(3)^2 - 12(3) + k$

$0 = 27 - 36 + k$

$k = 9$

$\frac{dy}{dx} = 3x^2 - 12x + 9$

$0 = 3x^2 - 12x + 9$

$x = 3$ and $x = 1$

$y = (3)^3 - 6(3)^2 + 9(3) - 5$

$= 27 - 54 + 27 - 5$

$= -5$ TP is $(3, -5)$

$y = (1)^3 - 6(1)^2 + 9(1) - 5$

$= 1 - 6 + 9 - 5$

$= 1$ TP is $(1, -1)$

Max is ~~(3, -5)~~ (1, -1) and Min is ~~(1, -1)~~ (3, -5)

r

E7

QUESTION THREE

ASSESSOR'S USE ONLY

(a) The gradient function of a curve is given by $\frac{dy}{dx} = -5x^4 + 6$.

The curve passes through (1,7).

Find the equation for y.

$y = -x^5 + 6x + c$

$7 = -(1)^5 + 6(1) + c$

$7 = -1 + 6 + c$

$7 = 5 + c$

$c = 2$ //

A

u

(b) Suppose that, at the start of a particular day, 1000 people were trading in a market, and that t days after the start of that day, the number of traders, N, can be modelled by

$N(t) = 1000 + 400t + 100t^2$.

How many days will it take for the rate of change of the number of traders to be 14 400 per day?

$N(t) = 1000 + 400t + 100t^2$

$N'(t) = 400 + 200t$

$14400 = 400 + 200t$

$14400 - 400 = 200t$

$14000 = 200t$

$t = \frac{14000}{200}$

$= 70$ A//

u

Question Three continues on the following page.

(c) A school is selling tickets for its drama production.

The revenue, $\$R$, from selling tickets for a price of $\$p$ each, can be modelled by the function

$$R(p) = 40p(29 - 2p)$$

Use calculus to find the maximum possible revenue (using this model).

ASSESSOR'S USE ONLY

$R(p) = 40p(29-2p)$

$= 1160p - 80p^2$

$R'(p) = -160p + 1160$

$0 = -160p + 1160$

$160p = 1160$

$p = \frac{1160}{160}$

$p = 7.25$

Max possible revenue is at 7.25 days

u

(d) A tangent to the graph of the function $y = -\frac{1}{3}x^3 + kx + 4$ at a certain point P, has gradient of –7 and intersects the graph again at (–6,64).

Use calculus to find the co-ordinates of the point P.

ASSESSOR'S USE ONLY

$\frac{dy}{dx} = -x^2 + k$

$\frac{dy}{dx} = -x^2 + 2$

$-7 = -x^2 + 2$

$-9 = -x^2$

$9 = x^2$

$\sqrt{9} = x$

$x = 3$

$64 = -\frac{1}{3}(-6)^3 + k(-6) + 4$

$64 = 72 + -6k + 4$

$60 = 72 + -6k$

$-12 = -6k$

$12 = 6k$

$k = 2$

$y = -\frac{1}{3}(3)^3 + 2 \times 3 + 4$

$y = -\frac{1}{3}(27) + 10$

$y = -9 + 10$

$y = 1$

coordinates of the point P are (3, 1)

A

t_1

E7

Excellence Exemplar 2018

Subject	Level 2 Mathematics and Statistics	Standard	91262	Total score	21

Q	Grade score	Annotation
1	E7	If the candidate had justified the area was a minimum in part (e) they may have scored E8.
2	E7	This is E7 because the answer to (c) was in context. A grade of E8 may have been obtained if the nature of the turning points in (e) were justified as was indicated in the question.
3	E7	This is E7 because in (d) only one value for x was found when solving the quadratic. Consequently, the opportunity to gain E8 by investigating the possibility of the other solution was missed.

2

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Excellence

TOTAL 24

ASSESSOR'S USE ONLY

ASSESSOR'S USE ONLY

QUESTION ONE

(a) A function f is given by $f(x) = x^3 - 6x + 2$.

Find the gradient of the graph of the function at the point where $x = 4$.

$f'(x) = 3x^2 - 6$

$f'(x) = 3(4)^2 - 6$

$f'(x) = 42$ //

u

(b) A rectangle is expanding in area so that at all times its length is three times its width.

Find the rate of change of the area of the rectangle with respect to its width when the area of the rectangle is 75 cm^2.

area = W x L

W = x

L = $3x$

(area) $y = 3x \times x$ $\quad$ $75 = 3x^2$

$= 3x^2$ $\quad$ $x = 5$ cm

$y'(x) = 6x$

$y'(5) = 6 \times 5$

$= 30$ cm^2/cm //

r

ASSESSOR'S USE ONLY

(c) The derivative of a function f is given by $f'(x) = -3x^2 + 12x$.

The graph of the function has a local minimum at the point (0,5).

Use calculus to find the value of the local maximum of the function.

$f(x) = -x^3 + 6x^2 + c$

$5 = -0^3 + 0 + c$

$c = 5$

$f(x) = -x^3 + 6x^2 + 5$

~~$f(x) = -x^3 + 6x^2 + c$~~

$f'(x) = -3x^2 + 12x$

$0 = -3x^2 + 12x$

$x = 0$ and $x = 4$

$f''(x) = -6x + 12$

$f''(0) = -6(0) + 12$

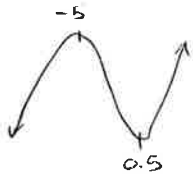
$f''=12$, $f'' > 0$ when $x = 0$ therefore it is a local minimum. $(0, 5)$

$f''(4) = -6(4) + 12$, $f'' = -12$

$f'' < 0$ when $x = 4$ therefore it is a local maximum at $(4, 37)$. //

r

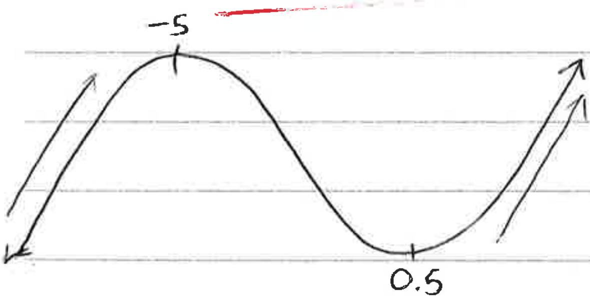
(d) Use calculus to find the values of x for which the graph of the function $f(x) = \frac{2}{3}x^3 + \frac{9}{2}x^2 - 5x - 18$ is increasing.



$f'(x) = 2x^2 + 9x - 5$

$0 = 2x^2 + 9x - 5$

$x = 0.5$ and $x = -5$



$f''(x) = 4x + 9$

$f''(0.5) = 4(0.5) + 9$

$= 11$

$f'' > 0$ when $x = 0.5$ ∴ it is a local minimum turning point.

$f''(x) = 4x + 9$

$f''(-5) = 4(-5) + 9$

$= -11$

$f'' < 0$ when $x = -5$ ∴ it is a local maximum turning point. //

consequently, $f(x)$ is increasing when $x < -5$ and when $x > 0.5$. //

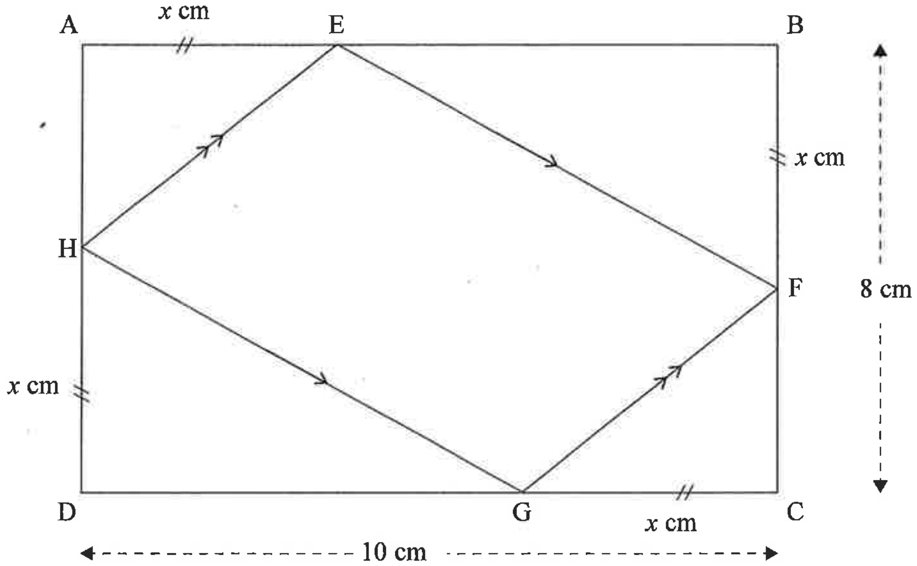
t

(e) A rectangle ABCD measures 10 cm by 8 cm. A parallelogram EFGH can be drawn inside the rectangle, as shown in the diagram below.

ASSESSOR'S USE ONLY

Suppose that the distance from each corner of the rectangle to the next vertex of the parallelogram, in a clockwise direction, is x cm.

That is, AE = BF = CG = DH = x.



Use calculus to find the smallest possible area that the parallelogram can have.

Justify that your answer is a minimum.

ABCD = $80\ cm^2$

$x(8-x) + x(10-x)$ = triangles area.

$80-(8x-x^2+10x-x^2)$ = parrallelgram area.

$80-18x+2x^2 = a$

$a' = -18+4x$

$0 = -18+4x$

$x = 4.5$

$a'' = 4$

$a'' > 0$ therefore the turning point is a minimum.

$a = 80-18(4.5)+2(4.5)^2$

$a = 39.5\ cm^2$

minimum area possible = $39.5\ cm^2$

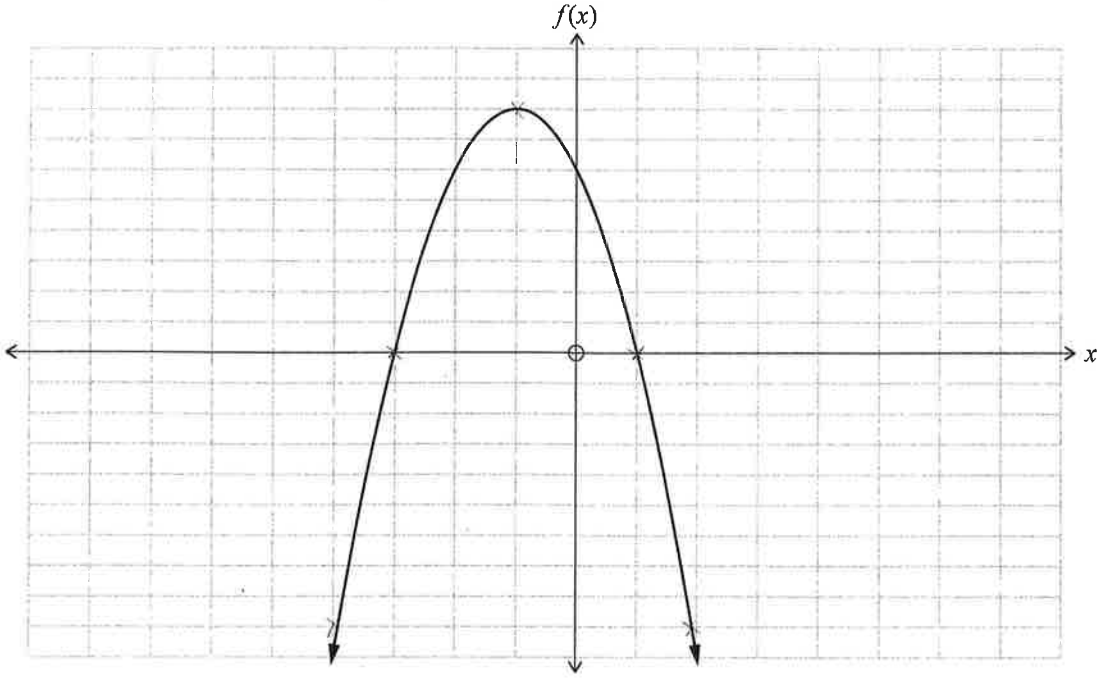
t

E8

ASSESSOR'S USE ONLY

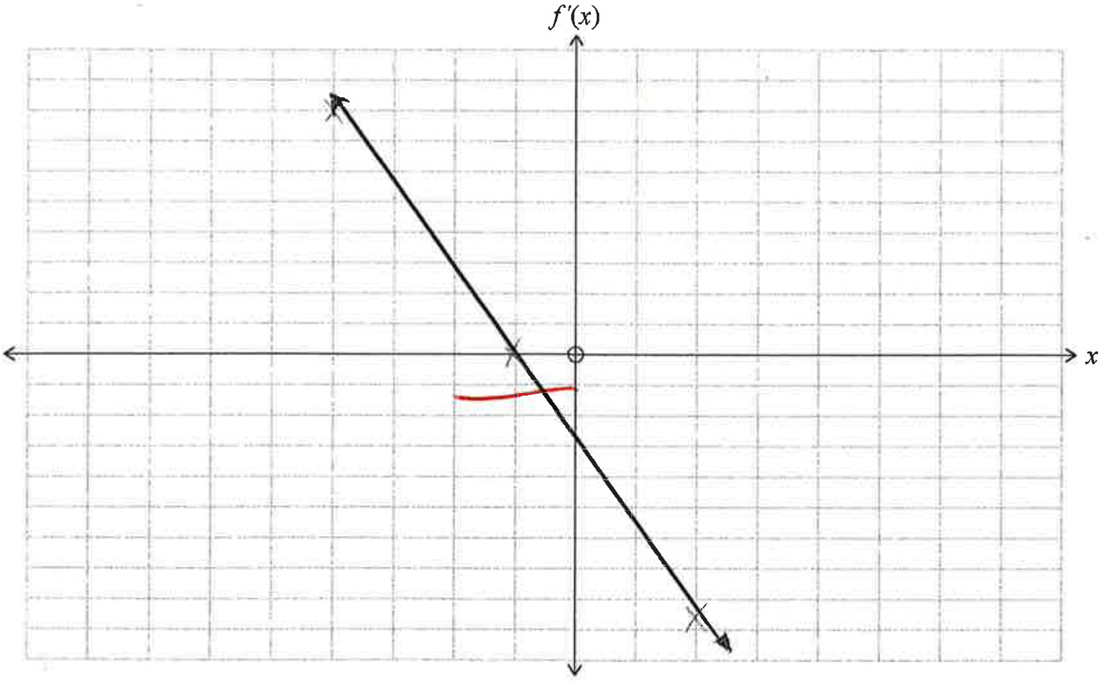
QUESTION TWO

(a) The graph of a function $y = f(x)$ is shown on the axes below.



Sketch the graph of the gradient function $y = f'(x)$ on the axes below.

Both sets of axes have the same scale.



4

If you need to redo this question part, use the grids on page 12.

ASSESSOR'S USE ONLY

(b) A skyrocket is projected into the air so that t seconds after it is launched, its height, h metres, above the ground is given by

$$h(t) = 39.2t - 4.9t^2.$$

What is the maximum height that the skyrocket will reach?

$h'(t) = 39.2 - 9.8t$
$0 = 39.2 - 9.8t$
$t = 4$ seconds

$h(4) = 39.2t - 4.9t^2$
$h(4) = 78.4$

$h''(t) = -9.8$
$h'' < 0 \therefore$ it is a local maximum.

The maximum height the sky rocket will reach is 78.4m, after four seconds of flight.

r

(c) Adam is operating his drone. It is moving in a straight line and t seconds after passing a tree its acceleration, a m s^{-2}, is given by

$$a(t) = 6 - 12t.$$

Two seconds after the drone passed the tree, its velocity was 20 m s^{-1}.

How far was the drone from the tree when its velocity was 20 m s^{-1}?

$a(t) = 6 - 12t$
$v(t) = 6t - 6t^2 + c$
$20 = 6(2) - 6(2)^2 + c$
$c = 32$
$v(t) = 6t - 6t^2 + 32$
$s(t) = 3t^2 - 2t^3 + 32t$

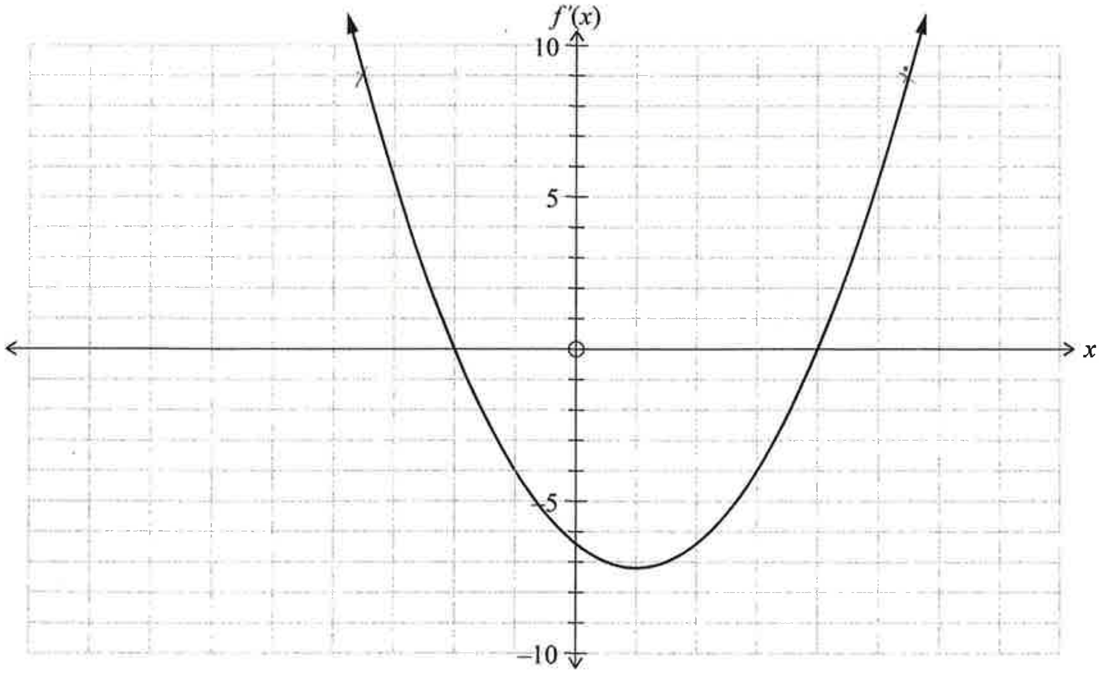
$s(2) = 3(2)^2 - 2(2)^3 + 32(2)$
$= 60$ metres

The drone was 60 metres from the tree when its velocity was $20ms^{-1}$

t

(d) The diagram below shows the graph of the gradient function $y = f'(x)$ of a function $y = f(x)$.

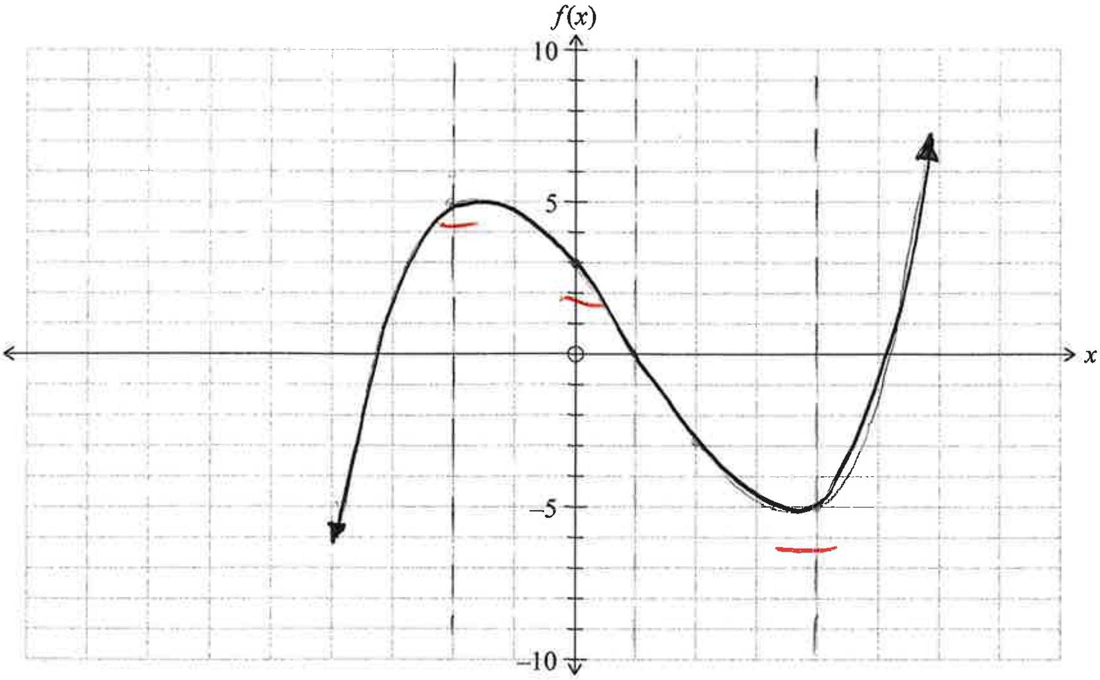
ASSESSOR'S USE ONLY



The graph of the function $y = f(x)$ passes through (0,3).

On the axes below sketch the graph of the function f.

Both sets of axes have the same scale.



If you need to redo this question part, use the grids on page 13.

ASSESSOR'S USE ONLY

(e) The graph of the function $y = x^3 - 6x^2 + kx - 5$ has a turning point at $x = 3$.

Use calculus methods to find the coordinates of both turning points.

Determine the nature of each turning point, justifying your answer.

$y' = 3x^2 - 12x + k$ ~~m~~

$0 = 3(3)^2 - 12(3) + k$

$k = 9$

$y' = 3x^2 - 12x + 9$

$0 = 3x^2 - 12x + 9$

$x = 3$ and $x = 1$

$y'' = 6x - 12$

$y''(3) = 6(3) - 12$

$= 6$

$y = x^3 - 6x^2 + 9x - 5$

$y = 3^3 - 6(3)^2 + 9(3) - 5$

$y = -5$

coordinates $(3, -5)$.

$y'' > 0$ when $x = 3$ therefore it is a minimum turning point at $(3, -5)$.

$y = x^3 - 6x^2 + 9x - 5$

$y = 1^3 - 6(1)^2 + 9(1) - 5$

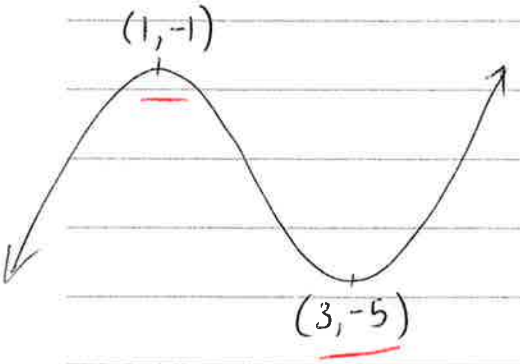
$y = -1$

coordinates $(1, -1)$

$y'' = 6x - 12$

$y''(1) = 6 - 12$

$= -6$



$y'' < 0$ when $x = 1$ therefore it is a maximum turning point at $(1, -1)$.

t

E8

QUESTION THREE

ASSESSOR'S USE ONLY

(a) The gradient function of a curve is given by $\frac{dy}{dx} = -5x^4 + 6$.

The curve passes through $(\overset{x}{1},\overset{y}{7})$.

Find the equation for y.

$y = -x^5 + 6x + c$

$7 = -1^5 + 6 + c$

$c = 2$

$y = -x^5 + 6x + 2$ //

u

(b) Suppose that, at the start of a particular day, 1000 people were trading in a market, and that t days after the start of that day, the number of traders, N, can be modelled by

$N(t) = 1000 + 400t + 100t^2$.

How many days will it take for the rate of change of the number of traders to be 14 400 per day?

$N'(t) = 400 + 200t$

$14{,}400 = 400 + 200t$

$t = 70$ days

It will take 70 days for the rate of change to be 14,400 traders per day. //

u

Question Three continues on the following page.

ASSESSOR'S USE ONLY

(c) A school is selling tickets for its drama production.

The revenue, $\$R$, from selling tickets for a price of $\$p$ each, can be modelled by the function

$$R(p) = 40p(29 - 2p)$$

Use calculus to find the maximum possible revenue (using this model).

$R(p) = 1160p - 80p^2$

$R'(p) = 1160 - 160p$

$0 = 1160 - 160p$

$\$p = 7.25$ each

$R(7.25) = 1160(7.25) - 80(7.25)^2$

$R(7.25) = \$4205$ revenue

$R'' = -160$ therefore $R'' < 0$

~~a~~ indicating it must be a maximum turning point.

max revenue $= \$4205$ //

(d) A tangent to the graph of the function $y = -\frac{1}{3}x^3 + kx + 4$ at a certain point P, has gradient of -7 and intersects the graph again at $(-6,64)$.

Use calculus to find the co-ordinates of the point P.

ASSESSOR'S USE ONLY

$\frac{dy}{dx} = -x^2 + k$

$-x^2 + k = -7$

~~64~~ Sub $(-6, 64)$ in y

$64\ \cancel{y} = -\frac{1}{3}(-6)^3 + kx + 4$

$64 = 72\ \cancel{A} - 6k + 4$

$64 = 76 - 6k$

$12 = 6k$

$k = 2$

$\frac{dy}{dx} = -x^2 + 2 = -7$

$x^2 = 9$

$x = \pm 3$

Tangent equation:

$y - 64 = -7(x + 6)$

$y = -7x + 22$

~~$-\frac{1}{3}x^3 + 2x + 4 = -7x + 22$~~

~~$-\frac{1}{3}x^3 + 9x - 18 = 0$~~

$y = -7x + 22$ sub in

i) $x = 3$ | ii) $x = -3$

$y = 1$ | $y = 43$

$y = -\frac{1}{3}x^3 + 2x + 4$ sub in

$x = 3$ | $y = -3$

$y = 1$ | $y = 7$

$\therefore x = 3$ as match in coordinates.

$P = (3, 1)$

t2

E8

Excellence Exemplar 2018

Subject	Level 2 Mathematics and Statistics	Standard	91262	Total score	24

Q	Grade score	Annotation
1	E8	There is a clearly stated correct interval for when the function is increasing in (d), and the minimum area has been found with justification in (e).
2	E8	This is E8 because in (c) the correct answer has been presented in context and in (e) the coordinates for max and min points have been given and each justified.
3	E8	The candidate in (d) has identified there are two possible x values and then shown that only one is valid for point P.