Assessment Schedule – 2018

Scholarship Calculus (93202)

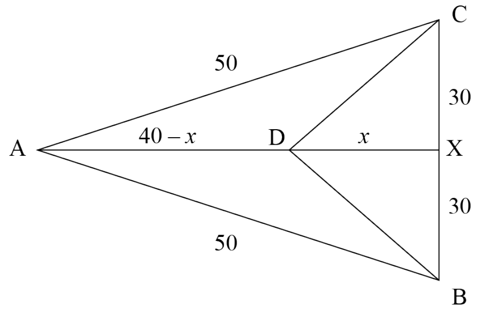
Evidence Statement

Q	Solution	1–4 No award	5–6 Schol	7–8 OS
ONE (a)	P: $(\cos\alpha,\sin\alpha)$; Q: $(\cos\beta,\sin\beta)$; R: $(\cos(\alpha-\beta),\sin(\alpha-\beta))$ $\angle ROA = \alpha - \beta = \angle POQ$, hence PQ = RA $PQ^2 = (\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2$ $= \cos^2\alpha - 2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta$ $= 2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$ Using the Cosine Rule $RA^2 = 1+1-2\cos(\alpha-\beta)$ Since PQ = RA, $1+1-2\cos(\alpha-\beta) = 2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$	Shows understanding of relevant mathematical concepts, and some progress towards solutions to problems.	Application of high-level mathematical knowledge and skills, leading to partial solutions to complex problems.	Application of high-level mathematical knowledge and skills, perception and insight / convincing communication shown in finding correct solutions to complex problems.
(b)	By the sine rule $\frac{\sin P}{OA} = \frac{\sin A}{OP}$ and so $\sin P = \frac{OA\sin A}{OP}$ OA and OP are of constant length i.e. sin P = $k$sin A where $k = \frac{OA}{OP}$ So, sin P is max and hence $\angle$APO is max when sin A = 1, i.e. $\angle A = \frac{\pi}{2}$, PA _|_ OA, with P on either side of OA.			

(c)	$\frac{\cos\theta}{1+\sin\theta}-\frac{\sin\theta}{1+\cos\theta}=\frac{\cos\theta+\cos^2\theta-\sin\theta-\sin^2\theta}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$ $=\frac{\cos\theta-\sin\theta+(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$ $=\frac{(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{1+\sin\theta+\cos\theta+\sin\theta\cos\theta}$ $=\frac{2(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{2+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}$ $=\frac{2(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{1+\sin^2\theta+\cos^2\theta+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}$ $=\frac{2(\cos\theta-\sin\theta)(1+\cos\theta+\sin\theta)}{(1+\sin\theta+\cos\theta)^2}$ $=\frac{2(\cos\theta-\sin\theta)}{1+\sin\theta+\cos\theta}$			

Q	**Solution**	**1–4 No award**	**5–6 Schol**	**7–8 OS**
TWO (a)(i)	Singularities (discontinuous at these points.): $x=\left\{-\frac{2}{3},\frac{3}{5},\frac{5\pm\sqrt{13}}{2}\right\}$	Shows understanding of relevant mathematical concepts, and some progress towards solutions to problems.	Application of high-level mathematical knowledge and skills, leading to partial solutions to complex problems.	Application of high-level mathematical knowledge and skills, perception and insight / convincing communication shown in finding correct solutions to complex problems.
(ii)	$\lim_{x\to\frac{1}{2}} f(x)=\left\{\frac{2\left[\frac{1}{2}\right]^2-1}{\left(3\times\frac{1}{2}+2\right)\left(5\times\frac{1}{2}-3\right)}-\frac{2-3\times\frac{1}{2}}{\left(\left[\frac{1}{2}\right]^2-5\times\frac{1}{2}+3\right)}\right\}=\frac{-8}{21}$			
(b)	Points of intersection: from $xy=\sqrt{2}$ write $y=\frac{\sqrt{2}}{x}$. Substituting into second equation $x^2-\left(\frac{\sqrt{2}}{x}\right)^2=1$ $x^4-2=x^2$ $(x^2+1)(x^2-2)=0$ So $(x^2-2)=0$, hence $x=\pm\sqrt{2}$ Curves intersect at $\left(\sqrt{2},1\right)$ and $\left(-\sqrt{2},-1\right)$ For the curve $xy=\sqrt{2}$, $\frac{dy}{dx}=\frac{\alpha^2}{2}$ Which evaluates to $\frac{-1}{\sqrt{2}}$ at both intersection points. For the curve $x^2-y^2=1$			

	$2x-2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{dy}{dx}=\frac{x}{y}$ Which evaluates to $\sqrt{2}$ at both intersection points. The product of the slopes is –1, so perpendicular. Note: At the point of intersection $\left(\alpha,\frac{\sqrt{2}}{\alpha}\right)$, For the curve $xy=\sqrt{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\sqrt{2}}{\alpha^2}$ For the curve $x^2-y^2=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\alpha^2}{\sqrt{2}}$			

(c)	 Let the length of DX be x km, as shown in the diagram. Length of the motorway $f(x)=40-x+2\sqrt{x^2+900}$ $f'(x)=-1+(x^2+900)^{-\frac{1}{2}}\times 2x$ When $f'(x)=0$, $2x=\sqrt{x^2+900}$ $4x^2=x^2+900$ $3x^2=900$ $x=\pm 10\sqrt{3}$ We consider only the positive result. $f''(x)=2(x^2+900)^{\frac{-1}{2}}-x(x^2+900)^{\frac{-3}{2}}2x$ $f''(10\sqrt{3})=2\times(300+900)^{\frac{-1}{2}}-600\times(300+900)^{\frac{-3}{2}}=0.04330$ $f''(10\sqrt{3})>0, so\ x=10\sqrt{3}$ gives a local minimum Minimum length of motorway is $40+30\sqrt{3}$ km			

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THREE (a)(i)	Let $u = 2x + 5$, then $x = \dfrac{u-5}{2}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}$ $\displaystyle\int_0^5 (2x-5)\sqrt{2x+5}\,\mathrm{d}x = \int_5^{15} (u-10)u^{\frac{1}{2}} \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ $\displaystyle= \int_5^{15} \left(u^{\frac{3}{2}} - 10u^{\frac{1}{2}}\right)\frac{1}{2}\mathrm{d}u$ $\displaystyle= \left[\frac{1}{5}u^{\frac{5}{2}} - \frac{10}{3}u^{\frac{3}{2}}\right]_5^{15} = 6.7225$	Shows understanding of relevant mathematical concepts, and some progress towards solutions to problems.	Application of high-level mathematical knowledge and skills, leading to partial solutions to complex problems.	Application of high-level mathematical knowledge and skills, perception and insight / convincing communication shown in finding correct solutions to complex problems.
(ii)	The definite integral cannot be evaluated since $\sqrt{2x-5}$ is not defined for $x < \dfrac{5}{2}$.			
(b)	Area between diagonal and curve: $\displaystyle\int_0^1 \left\{x - \left(\frac{\mathrm{B}-1}{\mathrm{B}}x^2 + \frac{1}{\mathrm{B}}x\right)\right\}\mathrm{d}x$ $\displaystyle= \int_0^1 \left\{\frac{\mathrm{B}-1}{\mathrm{B}}x - \frac{\mathrm{B}-1}{\mathrm{B}}x^2\right\}\mathrm{d}x$ $\displaystyle= \frac{\mathrm{B}-1}{\mathrm{B}}\int_0^1 \left\{x - x^2\right\}\mathrm{d}x$ $\displaystyle= \frac{\mathrm{B}-1}{\mathrm{B}}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \mathrm{d}x$ $\displaystyle= \frac{\mathrm{B}-1}{6\mathrm{B}}$ Area beneath diagonal: $\displaystyle\int_0^1 x\,\mathrm{d}x = \frac{1}{2}$			

	Coefficient of inequality $= \frac{B-1}{6B} \div \frac{1}{2} = \frac{B-1}{3B}$ To find B : $\frac{B-1}{3B} = \frac{20}{63}$ $B = 21$			

(c)	Volume of water flowing from the tank = volume of water flowing out of spout. Cross sectional area of tank × rate at which height is changing = cross sectional area spout × v water. $\frac{dV}{dt} = \pi\left(\frac{d}{2}\right)^2 v = \pi\left(\frac{d}{2}\right)^2 \sqrt{2gh}$ $\frac{dV}{dt} = \pi\left(\frac{D}{2}\right)^2 \frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{D^2}{d^2\sqrt{2g} \times \sqrt{h}}\mathrm{d}h = \mathrm{d}t$ $\frac{D^2}{d^2\sqrt{2g}}\int_{h_2}^{h_1}\frac{1}{\sqrt{h}}\mathrm{d}h = \left\|\int_{t_2}^{t_1}\mathrm{d}t\right\|$ $\frac{2D^2}{d^2\sqrt{2g}}\left[\sqrt{h}\right]_{h_2}^{h_1} = \left\|\left[t\right]_{t_2}^{t_1}\right\|$ $\left(\frac{D}{d}\right)^2 \times \sqrt{\frac{2}{g}} \times \left(\sqrt{h_1} - \sqrt{h_2}\right) = \|t_1 - t_2\|(= \Delta t)$			

Q	Solution	1-4 No award	5-6 Schol	7-8 OS
FOUR (a)	$\log_3(4-2x)+\log_3 x \le \log_3 9$ $\log_3(4x-2x^2) \le \log_3 9$ $4x-2x^2 \le 9$ $2x^2-4x+9 \ge 0$ Which is true for all $x \in \mathbb{R}$ However, for validity checking, require $4-2x>0$ and $x>0$ i.e. $\{x : 0<x<2\}$	Shows understanding of relevant mathematical concepts, and some progress towards solutions to problems.	Application of high-level mathematical knowledge and skills, leading to partial solutions to complex problems.	Application of high-level mathematical knowledge and skills, perception and insight / convincing communication shown in finding correct solutions to complex problems.
(b)	Circle one has centre (8,10) and radius 7. Circle two has centre (–4,5) and radius 6. The distance between their centres is $d=\sqrt{(8-(-4))^2+(10-5)^2}=13$. Since this is the sum of their radii, therefore the circles are tangential. Point of tangency: $P(x,y)$ divides the line joining their centres in the ratio 6:7. So $x=\frac{6\times 8+7\times -4}{6+7}=\frac{20}{13}, y=\frac{6\times 10+7\times 5}{6+7}=\frac{95}{13}$			

(c)	$(x+y)^2 - 2xy + 3(x+y) = 8$ $xy = 2 - 4(x+y)$ Substitute the second equation into the first: $(x+y)^2 - 2\big(2 - 4(x+y)\big) + 3(x+y) = 8$ $(x+y)^2 + 11(x+y) - 12 = 0$ $x+y=1$ or $x+y=-12$ When $x+y=1$ $xy = 2 - 4\times 1 = -2$ $\rightarrow x=2, y=-1$ or $x=-1, y=2$ When $x+y=1-12$ $xy = 2 - 4\times -12 = 50$ $\rightarrow x=-6+\mathrm{i}\sqrt{14}, y=-6-\mathrm{i}\sqrt{14}$ or $x=-6-\mathrm{i}\sqrt{14}, y=-6+\mathrm{i}\sqrt{14}$			

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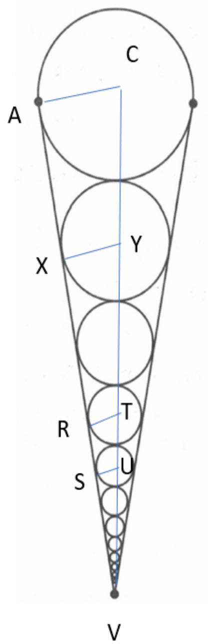
FIVE (a)	$z=\frac{a^2-bc+a(b+c)\mathrm{i}}{2a+(b+c)\mathrm{i}}\times\frac{2a-(b+c)\mathrm{i}}{2a-(b+c)\mathrm{i}}$ $=\frac{2a^3-2abc+2a^2(b+c)\mathrm{i}-a^2(b+c)\mathrm{i}+bc(b+c)\mathrm{i}+a(b+c)^2}{4a^2+(b+c)^2}$ $\mathrm{Re}(z)=\frac{2a^3-2abc+a(b+c)^2}{4a^2+(b+c)^2}$ $=\frac{2a^3+ab^2+ac^2}{4a^2+(b+c)^2}$	Shows understanding of relevant mathematical concepts, and some progress towards solutions to problems.	Application of high-level mathematical knowledge and skills, leading to partial solutions to complex problems.	Application of high-level mathematical knowledge and skills, perception and insight / convincing communication shown in finding correct solutions to complex problems.
(b)(i)	ΔACV and ΔXYV are similar. Both have a right angle (tangent radius) and they share a common angle at V. So, $\frac{\text{AC}}{\text{CV}}=\frac{\text{XY}}{\text{YV}}$ (1) $\text{XY}=r_2$ $\text{YV}=\text{CV}-\text{CY}=120-(20+r_2)$ $\text{YV}=100-r_2$ Substitute into Eq (1) $\frac{20}{120}=\frac{1}{6}=\frac{r_2}{100-r_2}$ $6r_2=100-r_2$ $r_2=\frac{100}{7}$ mm (14.285) 			

(ii)	Assume in the diagram above, we are looking at the radii r_n and r_{n+1} of spheres centre T and U respectively. Then, a similar argument follows as before. Triangles TRV and USV are both similar to triangle CAV. So: $\frac{RT}{TV} = \frac{SU}{UV} = \frac{AC}{CV} = \frac{1}{6}$ $\frac{r_n}{TV} = \frac{1}{6}$ and $6r_n = TV$. Also $6r_{n+1} = UV$ But $TV = TU + UV$ $6r_n = r_n + r_{n+1} + 6r_{n+1}$ from which $\frac{r_{n+1}}{r_n} = \frac{5}{7}$ We have a geometric sequence with $T_1 = 20$ and common ratio $r = \frac{5}{7}$. So the nth radius is given by $r_n = 20\left(\frac{5}{7}\right)^{n-1}$.			
(iii)	Volume of the spheres is given by: $V = \sum_{n=1}^{\infty} \frac{4}{3}\pi\left(20\left(\frac{5}{7}\right)^{n-1}\right)^3 = \frac{4}{3}\pi 20^3 \sum_{n=1}^{\infty}\left(\frac{5}{7}\right)^{3(n-1)}$ and since $r < 1$, the series converges. $S_\infty = \frac{a}{1-r}$ $V = \frac{4}{3}\pi 20^3 \times \left\{\frac{1}{1-\left(\frac{5}{7}\right)^3}\right\} = \frac{5\,488\,000\pi}{327}\ \text{mm}^3$			