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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2018
Calculus

9.30 a.m. Friday 9 November 2018
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

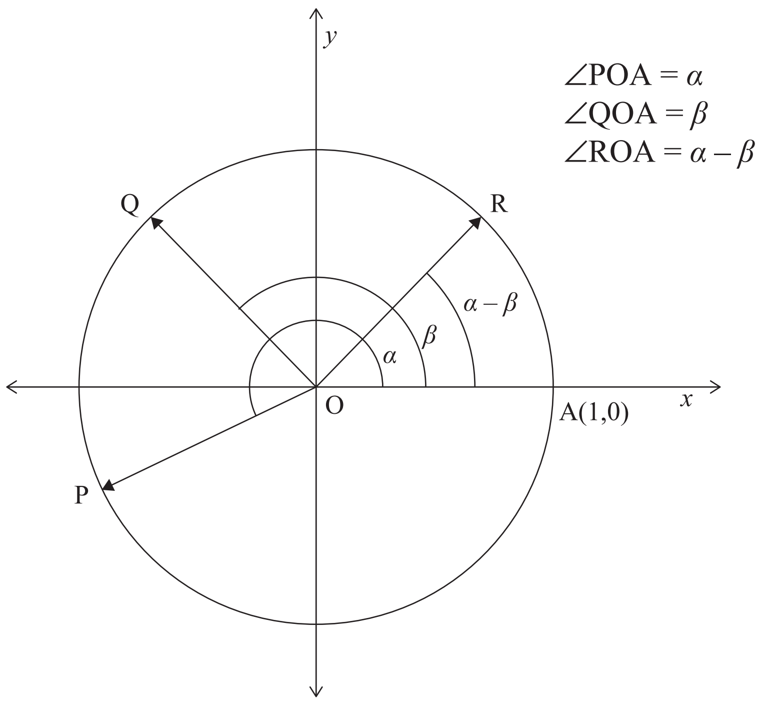
Check that this booklet has pages 2–7 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

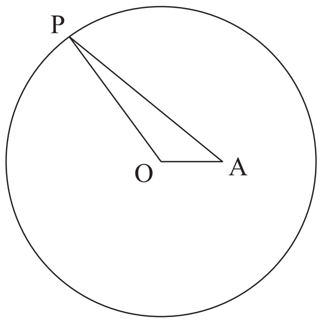
QUESTION ONE

(a) Use the unit circle shown below, or some other method, to prove from first principles:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$



(b) The diagram below shows a circle with centre O. There is a fixed point, A, inside the circle. P is a point on the circumference.



Where should P be located in relation to A so that ∠APO is a maximum?

Justify your answer.

(c) Prove the identity:

$$\frac{\cos\theta}{1+\sin\theta} - \frac{\sin\theta}{1+\cos\theta} = \frac{2(\cos\theta - \sin\theta)}{1+\sin\theta+\cos\theta}$$

QUESTION TWO

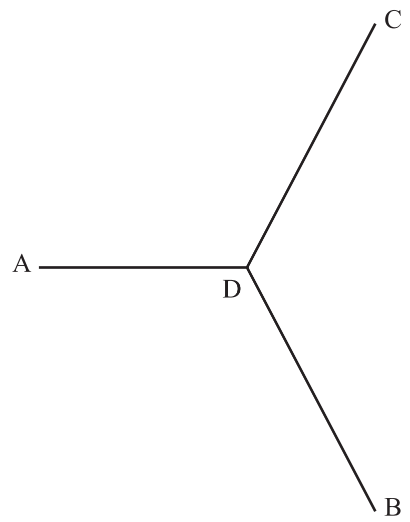
(a) Consider the function $f(x)=\dfrac{2x^2-1}{(3x+2)(5x-3)}-\dfrac{2-3x}{(x^2-5x+3)}$ where $x \in \mathbb{R}$.

(i) Give the exact value(s) for x for which the function f is discontinuous.

(ii) Find $\lim\limits_{x \to \frac{1}{2}} f(x)$.

(b) Show that the graphs of $xy=\sqrt{2}$ and $x^2-y^2=1$ have perpendicular tangents at their points of intersection.

(c) A roading system is to be designed so that three towns A, B, and C are connected as shown in the diagram below. The towns form an isosceles triangle with AB = AC = 50 km and BC = 60 km. The system is to be designed to minimise the total length of roading. The system is designed so that there is an intersection at a point D such that there are three straight sections of road: AD, DB, and DC. Sections DB and DC are to have the same length.



Use calculus to find the location of D that gives the minimum total length of roading.

Show that this length is $40+30\sqrt{3}$ km and show that it is indeed a minimum.

QUESTION THREE

(a) (i) Using a suitable substitution, or some other technique, evaluate the integral:

$$\int_0^5 (2x-5)\sqrt{2x+5}\,\mathrm{d}x$$

The result of any integration must be shown.

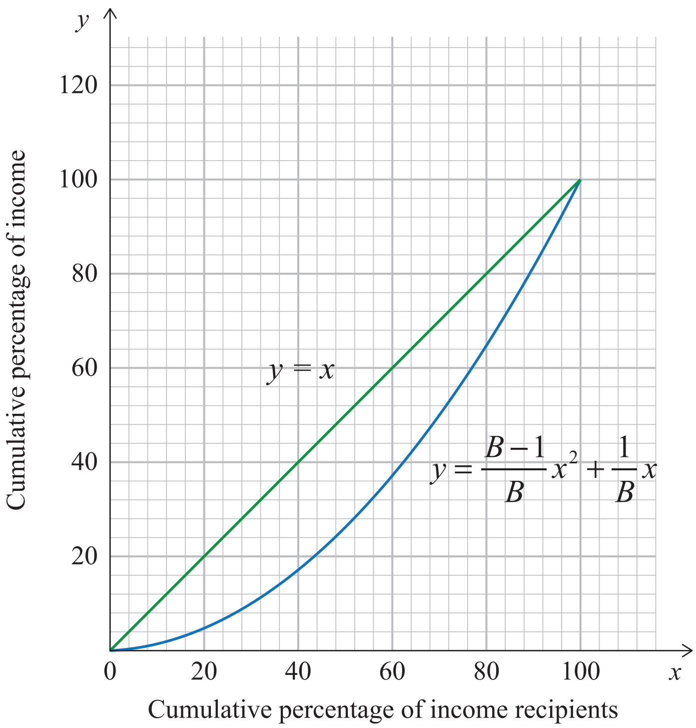
(ii) Explain why the method you used in (i) cannot be used to find the following:

$$\int_0^5 (2x+5)\sqrt{2x-5}\,\mathrm{d}x$$

(b) A Lorentz Curve is used to study the distribution of income. The diagram alongside shows a Lorentz Curve and the line $y = x$ where:

- x is the cumulative percentage of income recipients, ranked from poorest to richest
- y is the cumulative percentage of income.

The line $y = x$ represents equal distribution of income amongst all the recipients. Along this line, for example, 10% of the people will receive 10% of the total income. This line is referred to as the equality diagonal.



Suppose that the actual distribution of income is given by a Lorentz Curve of the form:

$$y = \frac{B-1}{B}x^2 + \frac{1}{B}x \qquad \text{where } B \text{ is a constant and } B \neq 1.$$

Using this curve, the income is not fairly distributed. This curve deviates from the equality diagonal. The degree of deviation from equality can be measured by the **coefficient of inequality**, which is defined as:

$$\frac{\text{area between the curve and the equality diagonal}}{\text{area beneath the equality diagonal}}$$

Suppose, after an income survey, it is found that the coefficient of inequality for a certain population is $\frac{20}{63}$.

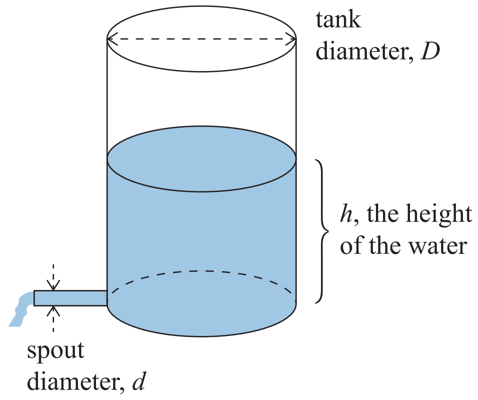
What, then, is the value of B for this population?

(c) Consider a regular cylindrical tank filled with water to height h. The water is being emptied freely through a rigid spout of fixed diameter d located at the base of the tank.

- The diameter of the tank is D.
- The diameter of the spout is d.
- Let h be the height of the water in the tank at any time.
- The water is passing out of the spout with velocity

  $$v = \sqrt{2gh}$$

  where g is the constant acceleration due to gravity.



Show that the time taken for the level of the water in the tank to drop from height h_1 to height h_2 is given by:

$$\left(\frac{D}{d}\right)^2 \times \sqrt{\frac{2}{g}} \times \left(\sqrt{h_1} - \sqrt{h_2}\right)$$

QUESTION FOUR

(a) Find the values of x for which: $\log_3(4 - 2x) + \log_3 x \leq 2$

(b) Show that the circles $x^2 + y^2 - 16x - 20y + 115 = 0$ and $x^2 + y^2 + 8x - 10y + 5 = 0$ are tangential to each other, and find the coordinates of the point of tangency.

(c) Find all solutions for x and y, both real and complex, in the system of simultaneous equations:

$$x^2 + y^2 + 3x + 3y = 8$$
$$xy + 4x + 4y = 2$$

QUESTION FIVE

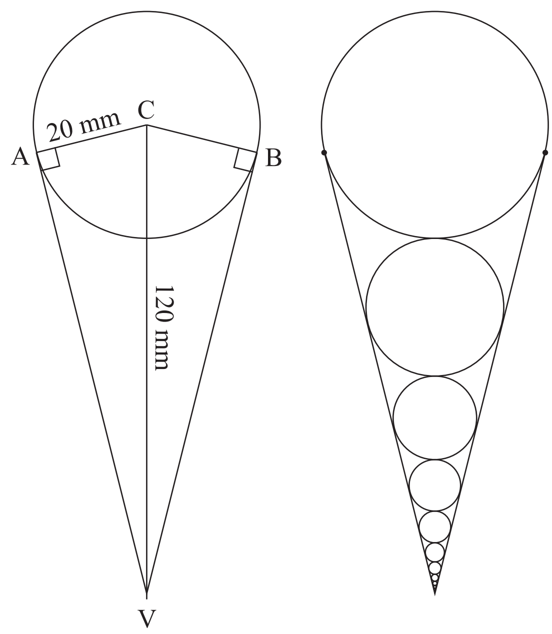
(a) Given $z = x + \mathrm{i}y$ and $z^{-1} = (a + \mathrm{i}b)^{-1} + (a + \mathrm{i}c)^{-1}$ where a, b and c are real, and $a \neq 0,\ b \neq 0,\ c \neq 0$

Find, in terms of a, b, and c, an expression for the real part of z.

(b) A spherical ball-bearing of radius 20 mm just fits inside a metal cone, touching at A and B. The centre C of the ball-bearing is 120 mm from the vertex of the cone, V.

Beneath the top ball-bearing is another spherical ball-bearing, and beneath that another, and so on, as shown in the diagram. After the first ball-bearing, each ball bearing:

- touches the one above it
- touches the sides of the cone.



(i) Find the radius, r_2, of the second ball-bearing.

(ii) Find an expression in terms of n for the radius of the nth ball-bearing, r_n.

(iii) Suppose this sequence of ball-bearings in the cone was infinite.

Show that the maximum volume of all the ball-bearings is $\dfrac{5.488 \times 10^6 \times \pi}{327}$ mm^3.

The following formulae may prove useful:

$$T_n = T_1 + (n-1)d \qquad T_n = T_1 r^{n-1} \qquad S_n = \frac{n}{2}\left[2T_1 + (n-1)d\right] \qquad S_n = T_1\left(\frac{1-r^n}{1-r}\right)$$