91578

3

SUPERVISOR'S USE ONLY

915780

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2015

91578 Apply differentiation methods in solving problems

2.00 p.m. Wednesday 25 November 2015
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–12 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 24

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QUESTION ONE

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(a) Differentiate $y = 6\tan(5x)$.

$30\sec^2(5x)$

(b) Find the gradient of the tangent to the function $y = (4x - 3x^2)^3$ at the point (1,1).

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = 3(4x-3x^2)^2(4-6x)$

when $x=1$

$3(4(1)-3(1)^2)^2(4-6)$

$\frac{dy}{dx} = -6$

gradient of normal $\frac{1}{6}$

(c) Find the values of x for which the function $f(x) = 8x - 3 + \frac{2}{x+1}$ is increasing.

You must use calculus and show any derivatives that you need to find when solving this problem.

$f'(x) = 8 - \frac{2}{(x+1)^2} > 0$

$\frac{2}{(x+1)^2} < 8$

$\frac{1}{4} < x^2 + 2x + 1$

$x^2 + 2x + \frac{3}{4} > 0$

$x = -\frac{1}{2}$ $\quad x = -\frac{3}{2}$

when $f(x)$ is increasing $\quad -\frac{3}{2} < x < -\frac{1}{2}$

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(d) For what value(s) of x is the tangent to the graph of the function $f(x)=\dfrac{x+4}{x(x-5)}$ parallel to the x-axis?

You must use calculus and show any derivatives that you need to find when solving this problem.

$f(x) = \dfrac{x+4}{x^2-5x} = (x+4)(x^2-5x)^{-1}$

$f'(x) = (x^2-5x)^{-1} - (x+4)(x^2-5x)^{-2}(2x-5) = 0$

$(x^2-5x) - (x+4)(2x-5) = 0$

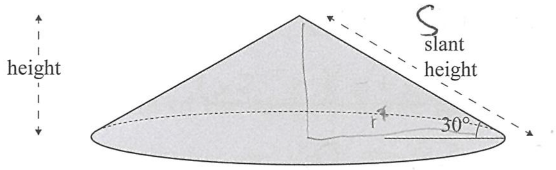
$x^2-5x = 2x^3-5x+8x-20$

$x^2-5x = 2x^3+3x-20$

$2x^3-x^2+8x-20=0$

$x = 1.68$

(e) Salt harvested at the Grassmere Saltworks forms a cone as it falls from a conveyor belt. The slant of the cone forms an angle of 30° with the horizontal. The conveyor belt delivers the salt at a rate of 2 m^3 of salt per minute.



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Find the rate at which the slant height is increasing when the radius of the cone is 10 m.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{r}{S} = \cos 30 \qquad \frac{r}{\cos 30} = S$

Pencil

$\frac{dV}{dt} = 2m^3 \qquad \cancel{\frac{dh}{dt}} \; \cancel{\frac{dh}{dV} \times \frac{dV}{dt}} \qquad \tan 30 = \frac{h}{r}$

$S = \cancel{\sqrt{\quad}} \; \frac{r}{\cos 30} \qquad h = \tan 30\, r$

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$

$V = \frac{1}{3}\pi r^2 (\tan 30 r) = \frac{\tan 30 \pi r^3}{3}$

$\frac{dS}{dr} = \frac{1}{\cos 30} \qquad \frac{dV}{dr} = \tan 30 \pi r^2$

$\frac{dS}{dt} = \frac{1}{\cos 30} \times \frac{1}{\tan 30 \pi r^2} \times 2\cancel{m^3}$

$= \frac{2}{\sin 30 \pi r^2}$

when $r = 10 \qquad \frac{dS}{dt} = \frac{2}{100 \sin 30 \pi}$

$= 0.013$ m per minute

t

t

E8

E8

QUESTION TWO

ASSESSOR'S USE ONLY

(a) Differentiate $f(x) = \sqrt[5]{x - 3x^2}$.

$f(x) = (x - 3x^2)^5$

$f'(x) = 5(x - 3x^2)^4(1 - 6x)$

(b) Find the gradient of the normal to the curve $y = x - \dfrac{16}{x}$ at the point where $x = 4$.

You must use calculus and show any derivatives that you need to find when solving this problem.

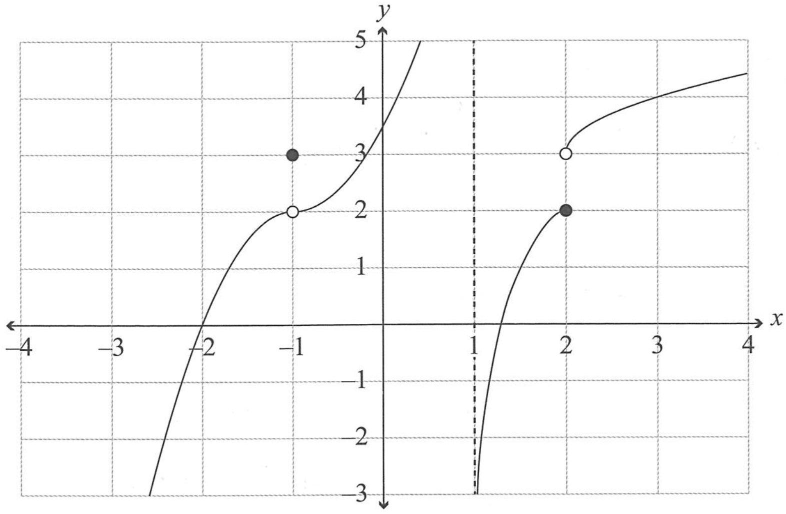
$\dfrac{dy}{dx} = 1 + \dfrac{16}{x^2}$

when $x = 4$ $\dfrac{dy}{dx} = 1 + 1 = 2$

gradient of normal $= -\dfrac{1}{2}$

(c) The graph below shows the function $y = f(x)$.

ASSESSOR'S USE ONLY



For the function above:

(i) Find the value(s) of x that meet the following conditions:

1. $f(x)$ is not defined: 1
2. $f(x)$ is not differentiable: $-1, 1, 2$
3. $f''(x) > 0$: $x < -1$, $1 < x < 2$

(ii) What is the value of $f(-1)$? 3
State clearly if the value does not exist.

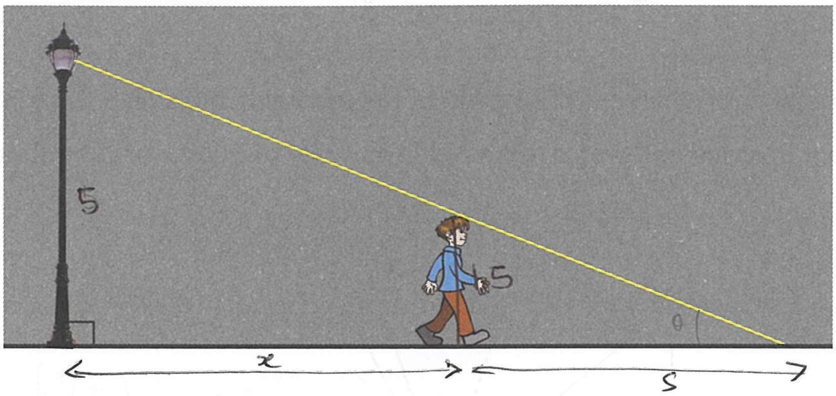
(iii) What is the value of $\lim_{x \to 2} f(x)$? does not exist
State clearly if the value does not exist.

(d) A street light is 5 m above the ground, which is flat.

A boy, who is 1.5 m tall, is walking away from the point directly below the streetlight at 2 metres per second.

At what rate is the length of his shadow changing when the boy is 8 m away from the point directly under the light?

You must use calculus and show any derivatives that you need to find when solving this problem.



x = length of boy to light

s = length of shadow

$\frac{dx}{dt} = 2 ms^{-1}$ $\qquad \frac{1.5}{5} = \frac{s}{s+x}$

$1.5s + 1.5x = 5s \qquad 1.5x = 3.5s \qquad s = \frac{3}{7}x$

$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$

$\frac{ds}{dx} = \frac{3}{7} \times 2 = \frac{6}{7} ms^{-1}$

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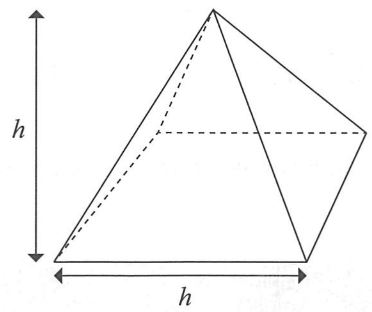
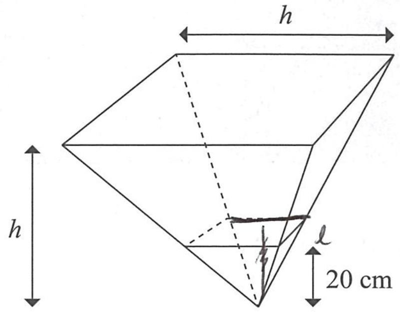
(e) A water container is constructed in the shape of a square-based pyramid. The height of the pyramid is the same as the length of each side of its base.



A vertical height of 20 cm is then cut off the top of the pyramid, and a new flat top added.

The pyramid is then inverted and water is poured in at a rate of 3000 cm^3 per minute.



Find the rate at which the surface area of the water is increasing when the depth of the water is 15 cm.

$$\text{Volume of pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

You must use calculus and show any derivatives that you need to find when solving this problem.

A = surface area $\quad \frac{20}{h} = \frac{l}{h} \quad l = 20 \quad$ volume of small pyramid = 2666.7

volume of ~~small~~ pyramid = $\frac{h^3}{3} - 2666.7 \qquad \frac{dV}{dt} = 3000\text{ cm}^3/\text{min}$

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt}$$

$A = h^2 \qquad \frac{dA}{dh} = 2h \qquad \frac{dV}{dh} = h^2$

$$\frac{dA}{dt} = 2h \times \frac{1}{h^2} \times 3000$$

$$= \frac{6000}{h}$$

when $h = 35 \qquad \frac{dA}{dt} = \frac{6000}{35} = 171.43$ cm^2 per min.

t

t

E8

E8

ASSESSOR'S USE ONLY

QUESTION THREE

(a) For what value(s) of x does the tangent to the graph of the function $f(x) = 5\ln(2x-3)$ have a gradient of 4?

You must use calculus and show any derivatives that you need to find when solving this problem.

$f'(x) = \frac{10}{2x-3} = 4$

$10 = 8x - 12$

$22 = 8x$

$x = {}^{8}/_{22}$

(b) If $f(x) = \frac{x}{e^{3x}}$, find the value(s) of x such that $f'(x) = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$f(x) = xe^{-3x}$

$f'(x) = e^{-3x} \cancel{+} -3xe^{-3x} = 0$

$e^{-3x}(1-3x) = 0$

$1 - 3x = 0$

$x = 1/3$

(c) A curve is defined parametrically by the equations $x = 3\cos t$ and $y = \sin 3t$.

Find the gradient of the normal to the curve at the point where $t = \frac{\pi}{4}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dt} = 3\cos 3t \qquad \frac{dx}{dt} = -3\cos t$

when $t = \frac{\pi}{4} \qquad \frac{dy}{dt} = -2.12 \qquad \frac{dx}{dt} = 2.12$

$\frac{dy}{dx} = \frac{-2.12}{2.12} = -1$

gradient of normal $= 1$

(d) The equation of motion of a particle is given by the differential equation

$$\frac{d^2x}{dt^2} = -k^2x$$

where x is the displacement of the particle from the origin at time t, and k is a positive constant.

(i) Show that $x = A\cos kt + B\sin kt$, where A and B are constants, is a solution of the equation of motion.

~~$x = A\cos kt + B\sin kt$~~

$\int \frac{1}{x}\, d^2x = \int -k^2\, dt^2$

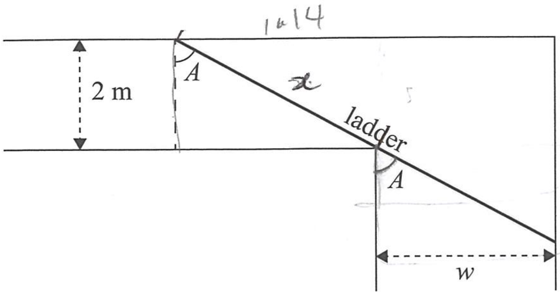
$\ln(x)\,dx = -k^2 t\, dt$

(ii) The particle was initially at the origin and moving with velocity 2k.

Find the values of A and B in the solution $x = A\cos kt + B\sin kt$.

(e) A corridor is 2 m wide.

At the end it turns 90° into another corridor.



2

$\tan A = \frac{w}{h}$

What is the minimum width, w, of the second corridor if a ladder of length 5 m can be carried horizontally around the corner?

You must use calculus and show any derivatives that you need to find when solving this problem.

$\cos A = \frac{2}{x}$ $\quad \frac{2}{\cos A} = x$

$\cancel{5-x}$ $\quad \cancel{w=}$ $\quad \frac{w}{5-x} = \sin A$

$w = \sin A\left(5 - \frac{2}{\cos A}\right)$

$= 5\sin A - \frac{2\sin A}{\cos A}$

$= 5\sin A - 2\tan A$

$\frac{dW}{dA} = 5\cos A - 2\sec^2 A = 0$

$5\cos A = \frac{2}{\cos^2 A}$

$5\cos^3 A = 2$

$A = 42.54°$

$W = 5\sin 42.54 - 2\tan 42.54$

$w = 1.55\text{ m}$

t

t

E8

E8

Extra paper if required.
Write the question number(s) if applicable.

QUESTION NUMBER

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91578

Annotated Exemplar Template

Excellence exemplar for 91578 2015			Total score	24
Q	**Grade score**	**Annotation**		
1	E8	1e The candidate has substituted to produce an expression for V in terms of only one variable, r. They have correctly formulated the related rates of change and substituted after the expression has been correctly differentiated. Some candidates differentiated the volume expression with out substituting for h and treated h as a constant. Some candidates substituted r=10, before differentiating.		
2	E8	2e, produced a correct expression for the volume of the truncated pyramid, correctly formulated the related rates of change and substituted after the expression has been correctly differentiated.		
3	E8	3e, formulated n expression for w in terms of only one variable, the angle, A. Differentiated the trig expression correctly and solved for w. All questions are well laid out and easy to follow.		