3

SUPERVISOR'S USE ONLY

91579

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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2015

91579 Apply integration methods in solving problems

2.00 p.m. Wednesday 25 November 2015
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply integration methods in solving problems.	Apply integration methods, using relational thinking, in solving problems.	Apply integration methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 24

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QUESTION ONE

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(a) Find $\int\left(\sqrt{x}+6\cos 2x\right)\mathrm{d}x$.

$= \int x^{\frac{1}{2}} + 6\cos 2x \, dx$

$= \frac{2}{3}x^{\frac{3}{2}} + 3\sin 2x + C$

(b) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{x}$, given that when $x = 1$, $y = 3$.

$y = 2\ln x + C$

$3 = 2\ln 1 + C$

$C = 3$

$y = 2\ln x + 3$

(c) If $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{e^{2x}}{4y}$ and $y = 4$ when $x = 0$, find the value of y when $x = 2$.

~~$4y\,dy = e^{2x}\,dx$~~

~~$2y^2 = 2e^{2x} + C$~~

~~$y = \sqrt{\frac{2e^{2x}+C}{2}}$~~

~~$4 = \sqrt{\frac{2+C}{2}}$~~

~~$C = 30$~~

~~$y = \sqrt{\frac{2e^4+30}{2}}$~~

~~$y = 8.34$ (2dp)~~

$4y\,dy = e^{2x}\,dx$

$2y^2 = \frac{e^{2x}}{2} + C$

$y^2 = \frac{e^{2x}}{4} + \frac{C}{2}$

$16 = \frac{1}{4} + \frac{C}{2}$

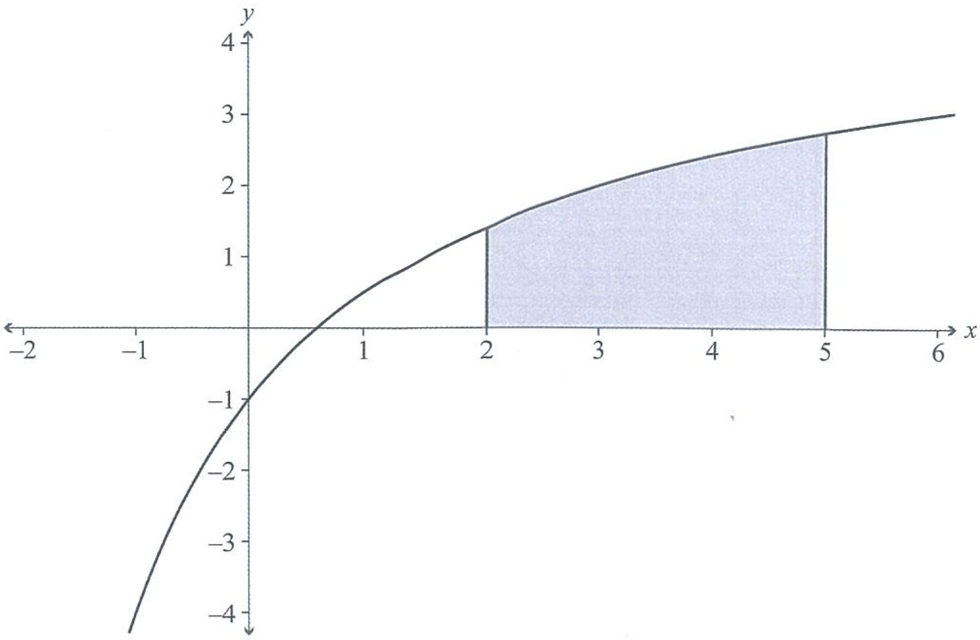
$C = 31.5$

$y^2 = \frac{e^4}{4} + \frac{31.5}{2}$

$y = 5.422$ (3dp)

(d) Use integration to find the area enclosed between the curve $y = \frac{5x-3}{x+3}$ and the lines $y = 0$, $x = 2$ and $x = 5$.

The area is shown shaded in the diagram below.



Show your working.

You must use calculus and give the results of any integration needed to solve this problem.

$$A = \int_2^5 \frac{5x-3}{x+1}\,dx$$

$$= \int_2^5 \frac{5(x+1)-8}{x+1}\,dx$$

$$= \int_2^5 5 - \frac{8}{x+1}\,dx$$

$$= \left[5x - 8\ln|x+1|\right]_2^5$$

$$= 25 - 8\ln 6 - 10 + 8\ln 3$$

$$= 9.455 \text{ units}^2 \text{ (3dp)}$$

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(e)

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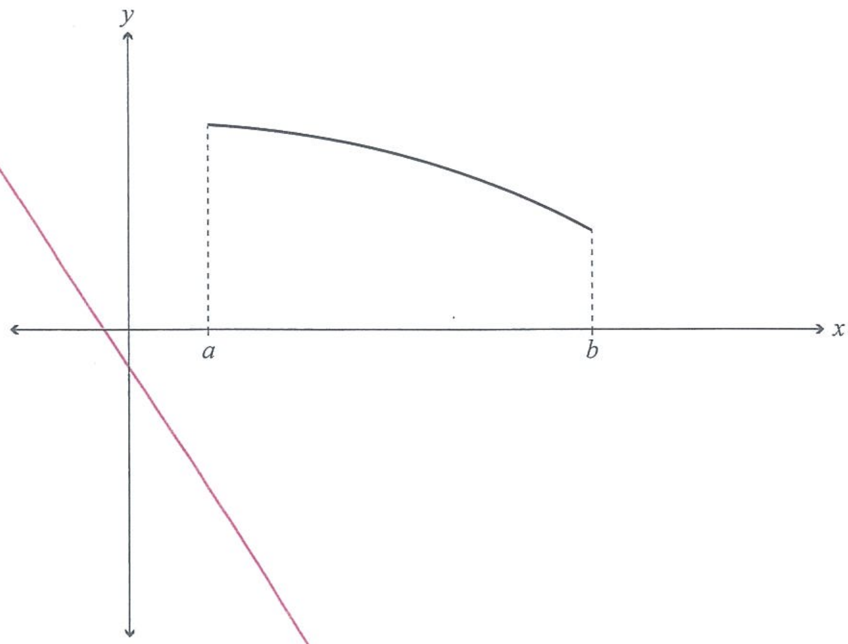
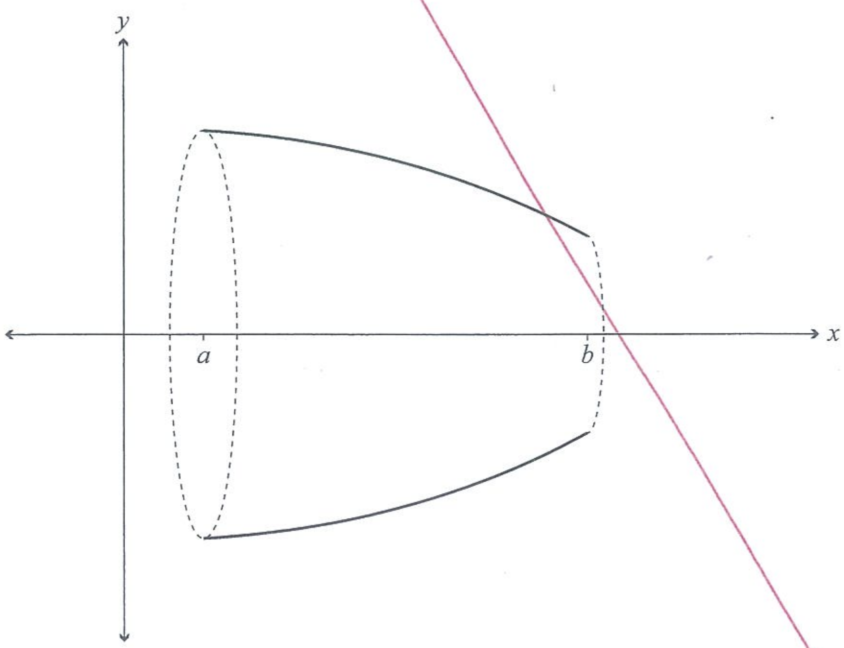
Consider the curve defined by the function $y = f(x)$, bounded by $x = a$ and $x = b$.



This portion of the curve is rotated around the x-axis, as shown below.



The volume created by this rotation is given by the formula

$$\text{Volume} = \pi\int_a^b \left(f(x)\right)^2 \mathrm{d}x$$

The graph below shows the function $y = \cos x$, between $x = 0$ and $x = \frac{\pi}{2}$, rotated around the x-axis.

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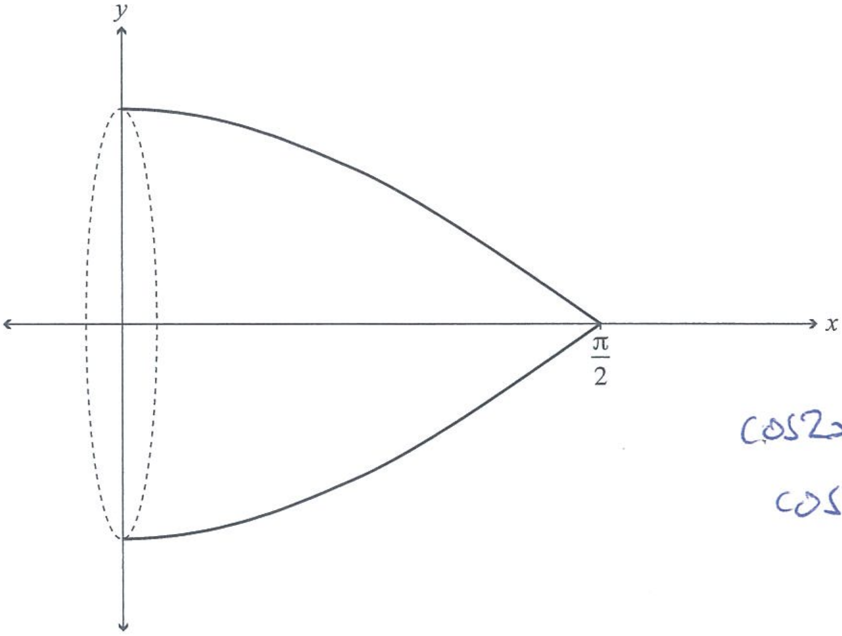


$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{\cos 2x + 1}{2}$

Find the volume created by this rotation.

You must use calculus and give the results of any integration needed to solve this problem.

$V = \pi \int_0^{\frac{\pi}{2}} (\cos x)^2 \, dx$

$\cancel{= \pi \left[\frac{(\cos x)^3}{3\sin x} \right]_0^{\frac{\pi}{2}}}$

$\cancel{= \pi \left(\frac{(\cos \frac{\pi}{2})^3}{3\sin \frac{\pi}{2}} - \right.}$

$V = \pi \int_0^{\frac{\pi}{2}} \frac{\cos 2x + 1}{2} \, dx$

$= \pi \left[\frac{\sin 2x}{4} + \frac{1}{2}x \right]_0^{\frac{\pi}{2}}$

$= \pi \left(\frac{\sin \pi}{4} + \frac{\pi}{4} \right)$

$= \boxed{\frac{\pi^2}{4} \text{ units}^3}$

E8

QUESTION TWO

ASSESSOR'S USE ONLY

(a) Find $\int\left(3-\frac{5}{x^2}\right)\mathrm{d}x$.

$3x + \frac{5}{x} + C$

(b) Use the values given in the table below to find an approximation to $\int_1^{2.5} f(x)\,\mathrm{d}x$, using the Trapezium Rule.

x	1	1.25	1.5	1.75	2	2.25	2.5
$f(x)$	0.3	0.7	1.65	1.9	2.35	1.7	1.1

(c) An object originally moving at a constant velocity suddenly starts to accelerate. From the start of the object's acceleration the motion of the object can be modelled by the differential equation

$$\frac{dv}{dt} = \frac{50t^2 - 80\sqrt{t}}{5\sqrt{t}} \quad \text{for } 0 \le t \le 20$$

where v is the velocity of the object in m s^{-1}

and t is the time in seconds after the object starts to accelerate.

If the original velocity of the object was 6 m s^{-1}, find the velocity of the object when t = 4.

You must use calculus and give the results of any integration needed to solve this problem.

$\frac{dv}{dt} = \cancel{50}\ 10t^{\frac{3}{2}} - 16$

$v = \cancel{1}4t^{\frac{5}{2}} - 16t + C$

$C = 6$

$v = 4(4)^{\frac{5}{2}} - 16(4) + 6$

$v = 70 \text{ ms}^{-1}$

ASSESSOR'S USE ONLY

(d) In the town of Clarkeville, the rate at which the population, P, of the town changes at any instant is proportional to the population of the town at that instant.

ASSESSOR'S USE ONLY

(i) Write a differential equation which models this situation.

$$\frac{dP}{dt} = KP$$

(ii) At the start of 2000, the population of the town was 12 000.

At the start of 2010, the population of the town was 16 000.

Solve the differential equation in (i) to find the population the town will have at the start of 2025.

You must use calculus and give the results of any integration needed to solve this problem.

$\ln P = Kt + C$ $\qquad e^C = A$

$P = e^{Kt+C}$

$P = Ae^{Kt}$ $\qquad t=0$ at year 2000

$A = 12{,}000$

$16000 = 12000e^{10K}$

$\frac{4}{3} = e^{10K}$

$10K = 0.2877$

$K = 0.02877$

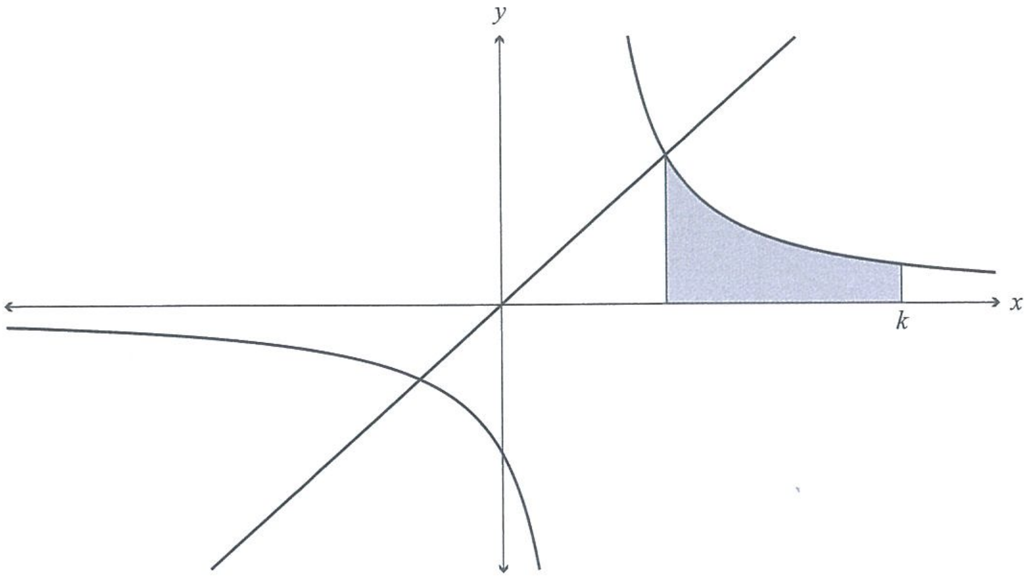
$P = 12000e^{(0.028755 \times 25)}$

$P = 24633.6$

$P = 24633$ people

ASSESSOR'S USE ONLY

(e) The graphs of $y = \dfrac{2}{x-1}$ and $y = x$ are shown on the axes below.



The shaded region has an area of 4 units squared.

Find the value of k.

You must use calculus and give the results of any integration needed to solve this problem.

$x = \dfrac{2}{x-1}$

$x^2 - x - 2 = 0$

$x = \boxed{2}$ or -1

$4 = \int_2^k \dfrac{2}{x-1}\,dx$

$4 = \left[2\ln|x-1|\right]_2^k$

$4 = 2\ln|k-1| - 2\ln 1$

$2 = \ln|k-1|$

$e^2 = k-1$

$k = e^2+1$

$k = 8.39$ (2dp)

f

E8

QUESTION THREE

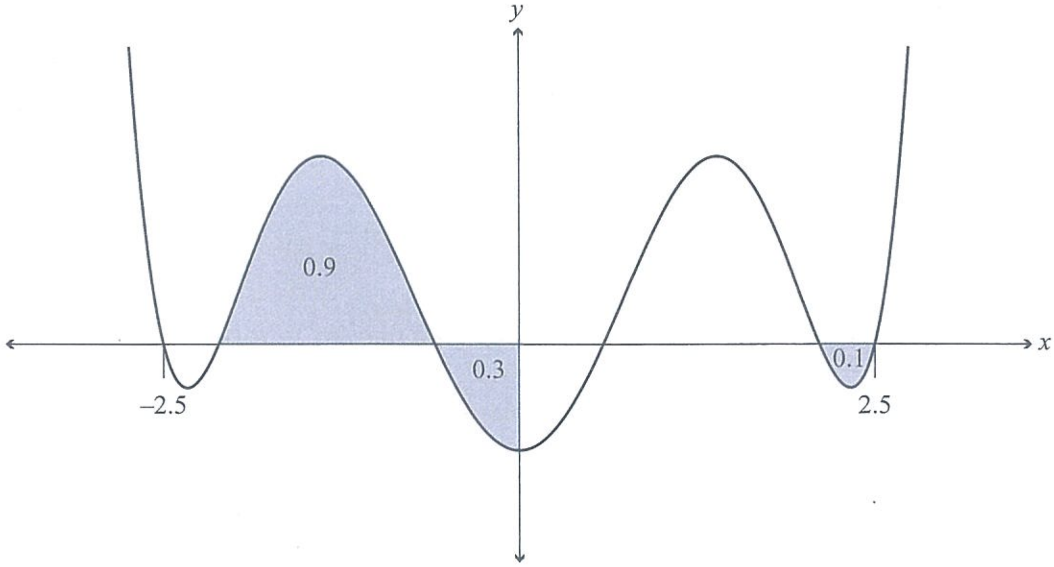
ASSESSOR'S USE ONLY

(a) Find $\int\left((x+4)^2+8e^{4x}\right)dx$.

$\frac{(x+4)^3}{3} + 2e^{4x} + c$

(b) The graph of the function $y = f(x)$ below is symmetrical about the y-axis.

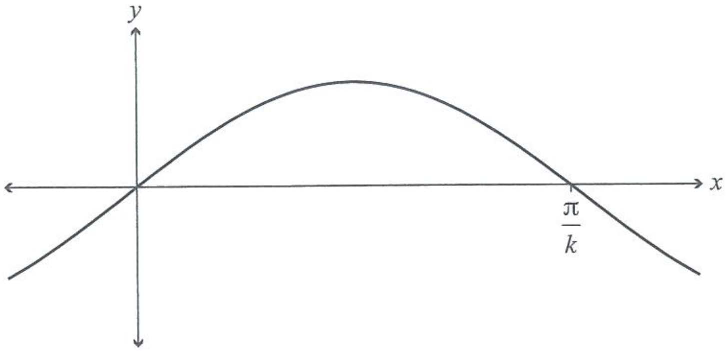
The areas of the shaded regions are given.



Find $\int_{-2.5}^{2.5} f(x)\,dx$.

$2(0.1+0.3+0.9) = 2.6$

(c) Find an expression in terms of k for the area bounded by the function $y = \sin kx$ and the x-axis, between $x = 0$ and $x = \frac{\pi}{k}$.



You must use calculus and give the results of any integration needed to solve this problem.

$A = \int_0^{\frac{\pi}{k}} \sin kx \, dx$

$= \left[-\frac{\cos kx}{k} \right]_0^{\frac{\pi}{k}}$

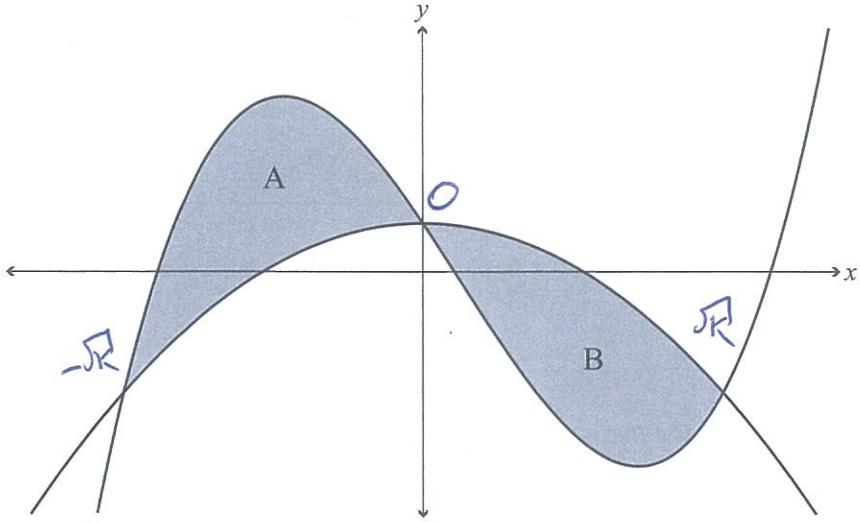
$A = \frac{-\cos \pi}{k} + \frac{\cos 0}{k}$

$A = \frac{2}{k}$

ASSESSOR'S USE ONLY

(d) The graphs of $f(x) = -x^2 + 2$ and $g(x) = x^3 - x^2 - kx + 2$ are shown below.

✱ The graphs intersect and create two closed regions, A and B.



Show that these two regions have the same area.

You must use calculus and give the results of any integration needed to solve this problem.

$$-x^2+2 = x^3-x^2-kx+2$$
$$0 = x^3-kx$$
$$x(x^2-k)=0$$
$$x=0 \quad \text{or} \quad x^2-k=0$$
$$x=\pm\sqrt{k}$$

$$\int_{-\sqrt{k}}^{0}(x^3-x^2-kx+2+x^2-2)\,dx = \int_{0}^{\sqrt{k}}(-x^2+2-x^3+x^2+kx-2)\,dx$$

$$\int_{-\sqrt{k}}^{0} x^3-kx\,dx = \int_{0}^{\sqrt{k}} -x^3+kx\,dx$$

$$\left[\frac{x^4}{4}-\frac{kx^2}{2}\right]_{-\sqrt{k}}^{0} = \left[\frac{-x^4}{4}+\frac{kx^2}{2}\right]_{0}^{\sqrt{k}}$$

$$\frac{0}{4}-\frac{k0}{2}-\frac{(-\sqrt{k})^4}{2}+\frac{k(-\sqrt{k})^2}{2} = \frac{-(\sqrt{k})^4}{4}+\frac{k^2}{2}-\frac{0^4}{2}+\frac{k0}{2}$$

$$\frac{-k^2}{2}+\frac{k^2}{2} = \frac{-k^2}{2}+\frac{k^2}{2}$$

LHS = RHS so areas are equal

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(e) An object starts from rest.

The object's acceleration is given by the formula $a = B\left(e^{kt}\right)^2$

where a is the acceleration of the object in m s^{-2}

and t is the time, in seconds, from when the object started moving.

Show that the time that it takes the object to reach velocity v_0 is

$$t = \frac{1}{2k}\ln\left(\frac{2v_0k + B}{B}\right)$$

You must use calculus and give the results of any integration needed to solve this problem.

$a = Be^{2kt}$

$\ln\left|\frac{a}{B}\right| = 2kt$

$t = \frac{\ln\left|\frac{a}{B}\right|}{2K}$

$V = \frac{Be^{2kt}}{2K} + C$ $\qquad C = \frac{-B}{2K}$

$V = \frac{B}{2K}e^{2kt} - \frac{B}{2K}$

$V = \frac{B}{2K}\left(e^{2kt} - 1\right)$

$V = \frac{B}{2K}\left(\frac{a}{B} - 1\right)$

$\frac{2kV}{B} + 1 = \frac{2kV + B}{B}$

$\frac{2kV}{B} + 1 = \frac{a}{B}$

$t = \frac{1}{2k}\ln\left(\frac{2kV_0}{B} + 1\right)$

$t = \frac{1}{2k}\ln\left(\frac{2kV_0 + B}{B}\right)$

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Extra paper if required.
Write the question number(s) if applicable.

QUESTION NUMBER

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