No part of the candidate evidence in this exemplar material may be presented in an external assessment for the purpose of gaining credits towards an NCEA qualification.

3

91578

915780

SUPERVISOR'S USE ONLY

NZQA

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2016

91578 Apply differentiation methods in solving problems

9.30 a.m. Wednesday 23 November 2016
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–15 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Achievement

TOTAL 10

ASSESSOR'S USE ONLY

ASSESSOR'S USE ONLY

QUESTION ONE

(a) Differentiate $y = 1 + x - \frac{1}{x} + \frac{1}{x^2}$.

$y = 1 + x - x^{-1} + x^{-2}$

$\frac{dy}{dx} = \cancel{8}\ 1 + x^{-2} - 2x^{-3}$

$= 1 + \frac{1}{x^2} - \frac{2}{x^3}$

(b) The height of the tide at a particular beach today is given by the function

$$h(t) = 0.8\sin\left(\frac{4\pi}{25}t + \frac{\pi}{2}\right)$$

where h is the height of water, in metres, relative to the mean sea level and t is the time in hours after midnight.

c2kiwi.blogspot.co.nz/2011/01/christchurch-wedding-stroll-on-beach.html

At what rate was the height of the tide changing at that beach at 9.00 a.m. today? 9 hrs

$h'(t) = 0.8 \cos\left(\frac{4\pi}{25}t + \frac{\pi}{2}\right) \times \left(\frac{4\pi}{25}\right)$

when $t = 9$

$h'(t) = 0.8 \cos\left(\frac{4\pi}{25} \times 9 + \frac{\pi}{2}\right) \times \left(\frac{4\pi}{25}\right)$

$= 0.395\text{ m}$ (3 dp)

ASSESSOR'S USE ONLY

(c) A curve is defined by the parametric equations

$x = 2\cos 2t$ and $y = \tan^2 t$. $\quad y = (\tan t)^2$

Find the gradient of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dx}{dt} = -2 \sin 2t \times 2$$

$$= -4 \sin 2t$$

$$\frac{dt}{dx} = \frac{1}{-4 \sin 2t}$$

$$\cancel{\frac{dy}{dt} = 2 \tan^3 t}$$

$$\cancel{y = \sec^2 t - 1}$$

$$\cancel{\frac{dy}{dt} = \tan t - t}$$

$$\cancel{\frac{dy}{dx} = (\tan t - t) \times \frac{1}{-4 \sin}}$$

$$\cancel{\frac{dy}{dx} = \frac{2(\tan t)^3 \sec^2 t}{-4 \sin 2t}}$$

$$\cancel{\text{when } t = \frac{\pi}{4}}$$

$$\cancel{\frac{dy}{dx} = \frac{2(\tan(\frac{\pi}{4}))^3 \times (\sec(\frac{\pi}{4}))^3}{-4 \sin 2(\frac{\pi}{4})}}$$

$$\cancel{= \frac{2 \times}{-4}}$$

$$\cancel{\frac{dy}{dt} = 2(\tan t)^3 \times \sec^2 t}$$

$$\cancel{\frac{dy}{dx}\ 7}$$

$$\cancel{y = \sec^2 t - 1}$$

$$\cancel{\frac{dy}{dt} = \tan t - t}$$

$$\cancel{\frac{dy}{dx} = \frac{\tan t - t}{-4 \sin 2t}}$$

$$\cancel{\text{when } t = \frac{\pi}{4}}$$

$$\cancel{\frac{dy}{dx} = \frac{\tan(\frac{\pi}{4}) - (\frac{\pi}{4})}{-4 \sin 2(\frac{\pi}{4})}}$$

A

Gradient = -1

RAWW

$$\frac{dy}{dt} = 2(\tan t)^3 \times (\sec t)^2$$

$$= 2(\tan t)^3 \times (\tan^2 t + 1)$$

$$\frac{dy}{dx} = \frac{2(\tan t)^3 \times (\tan^2 t + 1)}{-4 \sin 2t}$$

$$= \frac{2(\tan(\frac{\pi}{4}))^3 \times (\tan^2(\frac{\pi}{4}) + 1)}{-4 \sin 2(\frac{\pi}{4})}$$

$$= \frac{2 \times 2}{-4}$$

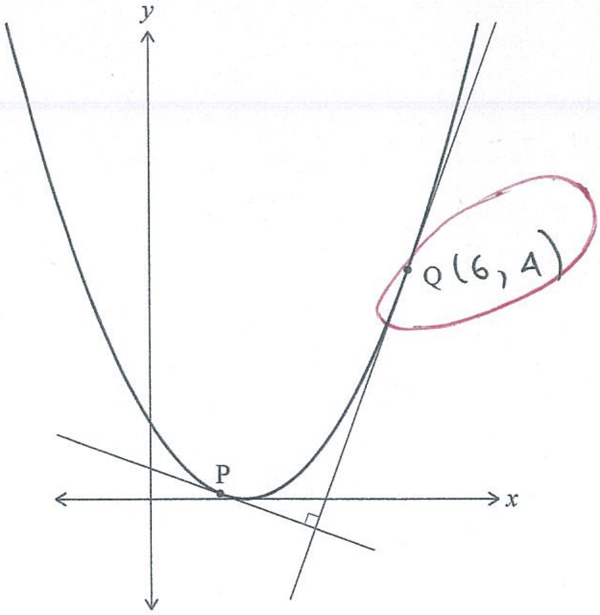
$$= \frac{4}{-4}$$

$$= -1$$

RAWW

(d) The tangents to the curve $y = \frac{1}{4}(x-2)^2$ at points P and Q are perpendicular.

ASSESSOR'S USE ONLY



Q is the point $(6, 4)$.

What is the x-coordinate of point P?

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \frac{1}{2}(x-2)$

when $x = 6$

$\frac{dy}{dx} = \frac{1}{2}(6-2)$

$= 2$

$y = 2x + 4$ (Q)

$-(x-k)^2$

$-2(x-k)$

(e) A curve is defined by the function $f(x) = e^{-(x-k)^2}$.

Find, in terms of k, the x-coordinate(s) for which $f''(x) = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$f'(x) = -2(x-k)e^{-(x-k)^2}$

$f''x = (-2(x-k))^2 e^{-(x-k)^2} = 0$

ASSESSOR'S USE ONLY

A4

QUESTION TWO

ASSESSOR'S USE ONLY

(a) Differentiate $f(x) = x\ln(3x-1)$.

$f'(x) = \frac{x}{3x-1} \times 3$

$= \frac{3x}{3x-1}$

(b) Find the gradient of the tangent to the function $y = \sqrt{2x-1}$ at the point $(5, 3)$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = (2x-1)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(2x-1)^{-1/2}$

~~$= \frac{1}{2}(2$~~

when $x = 5$

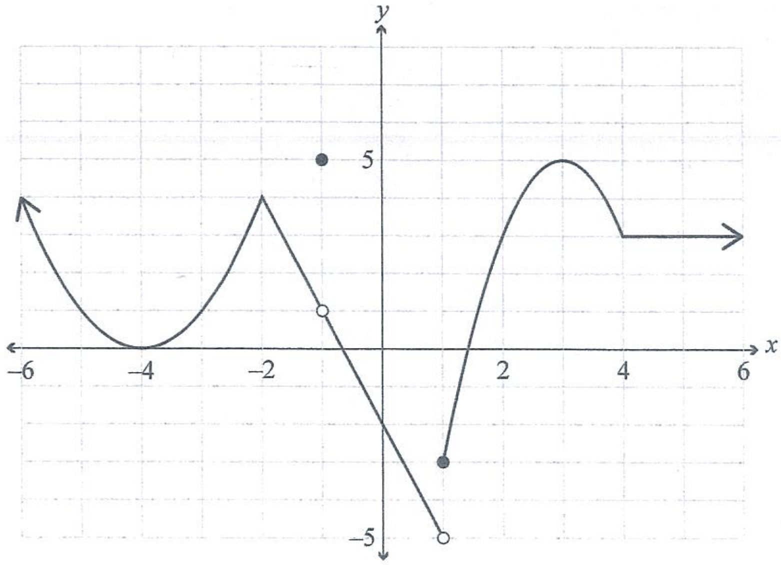
$\frac{dy}{dx} = \frac{1}{2}(2(5)-1)^{-1/2}$

$= \frac{1}{6}$

Gradient $= \frac{1}{6}$ at $(5, 3)$

ASSESSOR'S USE ONLY

(c) The graph below shows the function $y = f(x)$.



For the function $y = f(x)$ above:

(i) Find the value(s) of x that meet the following conditions:

1. f is not continuous: $x = -1, 1$
2. f is not differentiable: $x = -2, -1, 4$
3. $f'(x) = 0$: $x = -2$
4. $f''(x) < 0$: $1 < x < 4$

(ii) What is the value of $\lim\limits_{x \to -1} f(x)$?

State clearly if the value of the limit does not exist.

Limit does not exist.

$\frac{1}{3} - \frac{3}{?} = 1$

(d) A large spherical helium balloon is being inflated at a constant rate of 4800 cm^3 s^{-1}.

At what rate is the radius of the balloon increasing when the volume of the balloon is 288 000π cm^3?

You must use calculus and show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

$V = \frac{4}{3}\pi r^3$

$\cancel{\frac{dV}{dr} = \frac{12}{3}\pi r^2 \quad \frac{dr}{dV} = \frac{1}{4\pi r^2}}$

$= 4\pi r^2$

$\frac{dV}{dt} = 4800$

$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$\cancel{4\pi r^3 = 3V}$

$\cancel{r^3 = \frac{3V}{4\pi}}$

$\cancel{r = \sqrt[3]{\frac{3V}{4\pi}}}$

$\cancel{= \left(\frac{3V}{4\pi}\right)^{1/3}}$

$\cancel{\frac{dr}{dV} = \frac{1}{3}\left(\frac{3V}{4\pi}\right)^{-2/3} \times \left(\frac{3}{4\pi}\right)}$

$\cancel{\frac{dr}{dt} = 4800}$

$4\pi r^3 = 3V$

$r^3 = \frac{3V}{4\pi}$

$r = \left(\frac{3V}{4\pi}\right)^{1/3}$

$\frac{dr}{dV} = \frac{1}{3}\left(\frac{3V}{4\pi}\right)^{-2/3} \times \left(\frac{3}{4\pi}\right)$

$\cancel{\frac{dr}{dt} = 4800 \times \frac{1}{4\pi r^2}}$

$\cancel{= \frac{4800}{4\pi r^2}}$

$\frac{dr}{dt} = 4800 \times \left(\frac{1}{3}\left(\frac{3V}{4\pi}\right)^{-2/3} \times \left(\frac{3}{4\pi}\right)\right)$

when $V = 288\,000\pi$

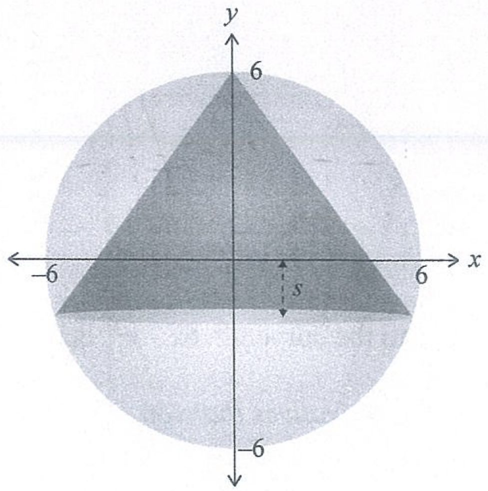
$\frac{dr}{dt} = 4800 \times \left(\frac{1}{3}\left(\frac{3(288000\pi)}{4\pi}\right)^{A} \times \left(\frac{3}{4\pi}\right)\right)$

$= 8250592 2 \cdot 5$

A

(e) A cone of height h and radius r is inscribed, as shown, inside a sphere of radius 6 cm.

ASSESSOR'S USE ONLY



The base of the cone is s cm below the x-axis.

Find the value of s which maximises the volume of the cone.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the volume you have found is a maximum.

$V = \frac{1}{3}\pi r^2 h$ (cone)

A3

QUESTION THREE

ASSESSOR'S USE ONLY

(a) Differentiate $f(x)=\sqrt[4]{3x+2}$.

$f(x) = (3x+2)^{1/4}$

$f'(x) = \frac{1}{4}(3x+2)^{-3/4} \times 3$

(b) Find the x-value at which a tangent to the curve $y = 6x - e^{3x}$ is parallel to the x-axis.

You must use calculus and show any derivatives that you need to find when solving this problem.

$f = 6x \quad f' = e^{3x} \quad g = 6 \quad g' = 3e^{3x}$

$\frac{dy}{dx} = 6x \times 3e^{3x} + e^{3x} \times 6$

$= 18xe^{3x} + 6e^{3x}$

$0 = 18xe^{3x} + 6e^{3x}$

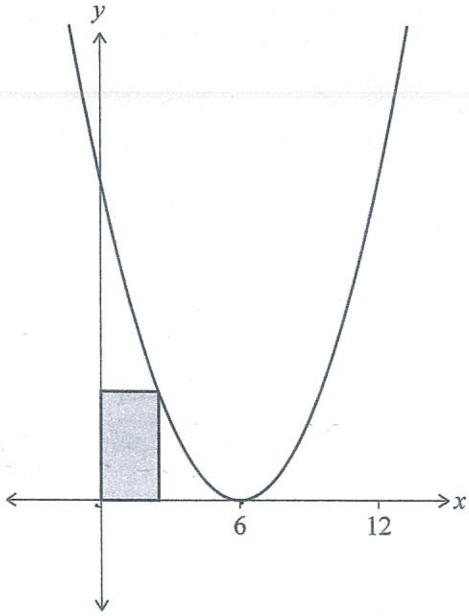
$18xe^{3x} = -6e^{3x}$

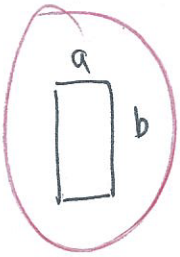
$3xe^{3x} = -e^{3x}$

$\cancel{3x \ln 3x = -\ln 3x}$

(c) A rectangle has one vertex at (0,0) and the opposite vertex on the curve $y = (x-6)^2$, where $0 < x < 6$, as shown on the graph below.

ASSESSOR'S USE ONLY





Find the maximum possible area of the rectangle.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the area you have found is a maximum.

$\frac{dy}{dx} = 2(x-6)$

Area = ab

ASSESSOR'S USE ONLY

(d) If $y = \frac{e^x}{\sin x}$, show that $\frac{dy}{dx} = y(1 - \cot x)$.

$g^2 = \sin^2 x$

$f = e^x \quad f' = e^x \quad g = \sin x \quad g' = \cos x$

$$\frac{dy}{dx} = \frac{e^x \times \sin x - e^x \times \cos x}{\sin^2 x}$$

$$= \frac{\sin x\, e^x - \cos x\, e^x}{\sin^2 x}$$

$$= \frac{\cancel{\sin x}\, e^x}{\sin^{\cancel{2}} x} - \frac{\cos x\, e^x}{\sin^2 x}$$

$$= \frac{e^x}{\sin x} - \frac{\cos x\, e^x}{\sin^2 x}$$

$$\cancel{= \left(\frac{e^x}{\sin x} - \left(\frac{\cos x\, e^x}{\sin x} \times \frac{1}{\sin x}\right)\right)}$$

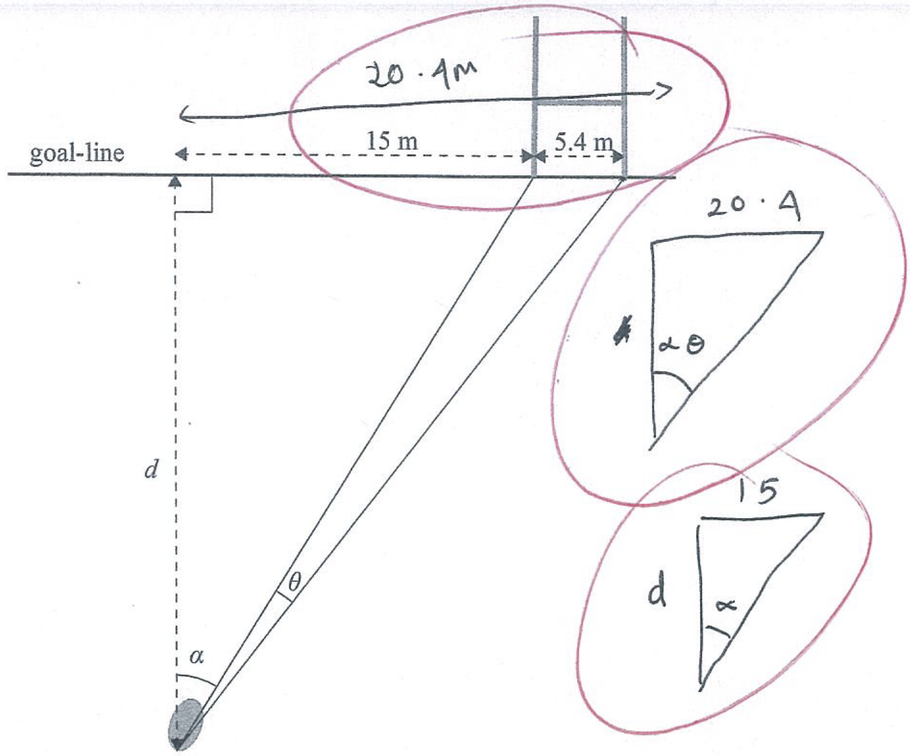
$$\cancel{= \frac{e^x}{\sin x} - (co}$$

A

(e) In a rugby game, a try is scored 15 m from the left-hand goal-post. The conversion kick is taken at some point on the line perpendicular to the goal-line from the point where the try was scored, as shown in the diagram below.

ASSESSOR'S USE ONLY

The ball needs to pass between the goal-posts, which are 5.4 m apart.



Find the distance d from the goal-line that the conversion kick should be taken from in order to maximise the angle θ between the lines from the ball to the goal-posts.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the angle you have found is a maximum.

~~$\frac{20.4}{d} = \frac{15}{d}$~~

~~$20.4 = \frac{15d}{d}$~~

~~$d = \tan\alpha$~~

$\tan\alpha = \frac{15}{d}$

~~$\alpha = \tan^{-1}\left(\frac{15}{d}\right)$~~

$d = \frac{15}{\tan\alpha}$

$\frac{dd}{d\alpha} = 15(\tan\alpha)^{-1}$

$= -15(\tan\alpha)^{-2} \times \sec^2\alpha$

$= \frac{-15\sec^2\alpha}{\tan^2\alpha}$

~~$d\tan\alpha = 15$~~

~~$d\tan\alpha - 15 = 0$~~

~~$d\sec^2\alpha = 0$~~

A3

Achievement exemplar 2016

Subject:	**Differentiation**	**Standard:**	**91578**	**Total score:**	**10**

Q	**Grade score**	**Annotation**
1	A4	This question provides evidence for A4 because the candidate has been able to demonstrate an ability to differentiate a variety of functions, some involving the chain rule. The functions correctly differentiated involved negative indices, trigonometric and exponential functions. This candidate was not able to achieve an M5 because they were not able to find the derivative of $y = tan^2 t$ and this was a common challenge for many of the candidates. The final result of -1 was correct but found with incorrect working.
2	A3	This question provides evidence for A3 because the candidate was able to locate all the points on the graph of the piecewise function where *f* was not continuous and where the graph of *f* was concave down. This candidate was also able to find a correct expression for $\frac{dr}{dt}$ when they were attempting to calculate the rate at which the radius of the balloon was increasing. They were not able to achieve an A4 because they were not able to apply the product rule to find the derivative of $f(x) = x\ln(3x-1)$ nor able to apply the chain rule to find the derivative of $y = \sqrt{2x-1}$.
3	A3	This question provides evidence for A3 because the candidate was able to find the derivative of a radical function involving a fractional power and the chain rule. They were also able to apply the quotient rule to find the derivative of the function: $y = \frac{e^x}{\sin x}$. This candidate was not able to achieve an A4 because they misread the function in part b as a product instead of a difference of functions and attempted to differentiate it using the product rule.