91578

3

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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2016

91578 Apply differentiation methods in solving problems

9.30 a.m. Wednesday 23 November 2016
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–15 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 21

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QUESTION ONE

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(a) Differentiate $y = 1 + x - \frac{1}{x} + \frac{1}{x^2}$.

$\cancel{\frac{dy}{dx} \Rightarrow}$ $y = 1 + x - x^{-1} + x^{-2}$.

$\frac{dy}{dx} = 1 + x^{-2} - 2x^{-3}$.

(b) The height of the tide at a particular beach today is given by the function

$$h(t) = 0.8 \sin\left(\frac{4\pi}{25}t + \frac{\pi}{2}\right)$$

where h is the height of water, in metres, relative to the mean sea level and t is the time in hours after midnight.

c2kiwi.blogspot.co.nz/2011/01/christchurch-wedding-stroll-on-beach.html

At what rate was the height of the tide changing at that beach at 9.00 a.m. today?

$h'(t) = 0.8 \times \frac{4\pi}{25} \cdot \cos\left(\frac{4\pi}{25}t + \frac{\pi}{2}\right)$

$= \frac{4}{5} \times \frac{4}{25}\pi \times \cos\left(\frac{4\pi}{25}t + \frac{\pi}{2}\right)$

when $t = 9$, $h'(t) = \frac{4}{5} \times \frac{4}{25}\pi \times \cos\left(\frac{36\pi}{25} + \frac{\pi}{2}\right)$

$= 0.395 \, m h^{-1}$

(c) A curve is defined by the parametric equations

$x = 2\cos 2t$ and $y = \tan^2 t$.

Find the gradient of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$(\cancel{\tan t})^2$. $y = u^2$

$\frac{dy}{du} = 2u$. $\frac{du}{dx} = \sec^2 x$.

$\sin = \frac{\sqrt{2}}{2}$

$\cos = \frac{\sqrt{2}}{2}$

$\frac{dx}{dt} = -4\sin 2t$.

$\frac{dy}{dt} = 2\sec^2 t \cdot \tan t$.

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-4\sin 2t}{2\sec^2 t \cdot \tan t}$.

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{-4\sin\frac{1}{2}\pi}{2\sec^2\frac{\pi}{4} \cdot \tan\frac{\pi}{4}}$

$= \frac{-4}{2 \cdot 2 \cdot 1}$

$= -1$.

nc

$g = 1$.

The gradient of the tangent to the curve is -1.

(d) The tangents to the curve $y = \frac{1}{4}(x-2)^2$ at points P and Q are perpendicular.

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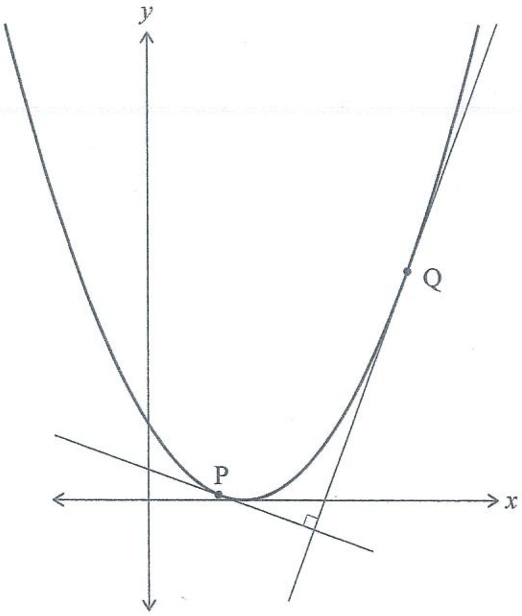

Q is the point $(6, 4)$.

What is the x-coordinate of point P?

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \frac{1}{2}(x-2) = \frac{1}{2}x - 1.$

when $x=6$, $\frac{dy}{dx} = \frac{1}{2}\times 6 - 1 = 2.$

$y = 2x + b.$

$4 = 12 + b$

$b = -8.$

$y = 2x - 8.$

P: $y = -\frac{1}{2}x + b.$

$-\frac{1}{2}x + b = \frac{1}{4}(x-2)^2.$

$-\frac{1}{2}x + b = \frac{1}{4}(x^2 - 4x + 4)$

$-\frac{1}{2}x + b = \frac{1}{4}x^2 - x + 1.$

$\frac{1}{4}x^2 - \frac{1}{2}x + 1 - b = 0.$

$\Delta = 0.$

$\frac{1}{4} - 4 \cdot \frac{1}{4} \cdot (1-b) = 0.$

$\frac{1}{4} - 1 + b = 0.$

$b = \frac{3}{4}.$

P: $y = -\frac{1}{2}x + \frac{3}{4}.$

$\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} = 0.$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1.$

The x-coordinate of P is 1.

(e) A curve is defined by the function $f(x) = e^{-(x-k)^2}$.

Find, in terms of k, the x-coordinate(s) for which $f''(x) = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

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~~$f'(x) = (x-k)^2 \cdot e^{(x-k)^2}$~~ $f(x) = e^{-x^2+2kx-k^2}$.

$f'(x) = (-2x+2k) \cdot e^{-x^2+2kx-k^2}$.

$f''(x) = (-2x+2k)^2 \cdot e^{-x^2+2kx-k^2}$.

$(-2x+2k)^2 \cdot e^{-x^2+2kx-k^2} = 0$.

$(-2x+2k)^2 = 0$.

$-2x+2k=0$.

$x=k$.

M5

QUESTION TWO

(a) Differentiate $f(x) = x\ln(3x - 1)$.

$f(x) = x \cdot \ln(3x-1)$.

$f = x,\ f' = 1.$ $\quad g = \ln(3x-1),\ g' = \frac{3}{3x-1}$

$f(x) = x \cdot \frac{3}{3x-1} + \ln(3x-1).$

(b) Find the gradient of the tangent to the function $y = \sqrt{2x-1}$ at the point $(5,3)$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = (2x-1)^{\frac{1}{2}}$.

$\frac{dy}{dx} = \frac{1}{2} \cdot 2 \cdot (2x-1)^{-\frac{1}{2}}$

$= (2x-1)^{\frac{1}{2}}$

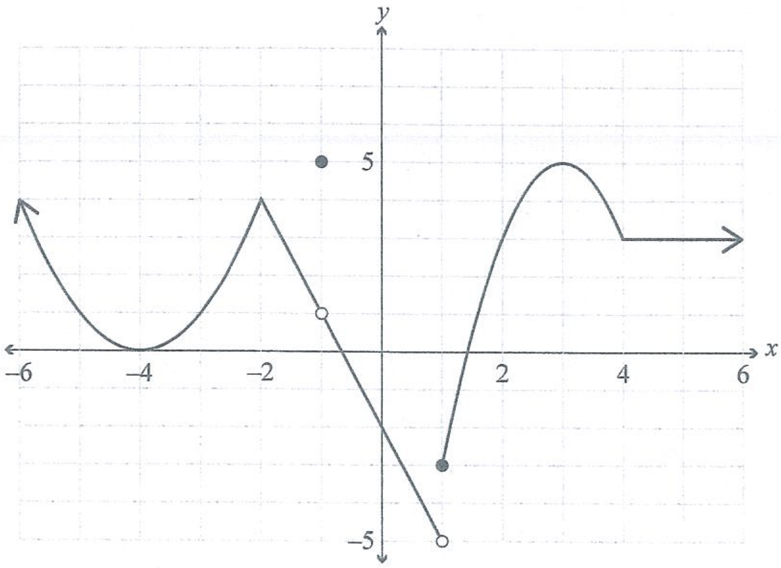
At the point $(5,3)$

$\frac{dy}{dx} = 9^{\frac{1}{2}} = 3$

$g = -\frac{1}{3}$ The gradient of the tangent is $-\frac{1}{3}$

(c) The graph below shows the function $y = f(x)$.

ASSESSOR'S USE ONLY



For the function $y = f(x)$ above:

(i) Find the value(s) of x that meet the following conditions:

1. f is not continuous: $x = -1,\ x = 1$
2. f is not differentiable: $x = -1,\ x = 1,\ x = -2$
3. $f'(x) = 0$: $x = -4,$
4. $f''(x) < 0$: $x < -4,\quad 1 < x < 3$

(ii) What is the value of $\lim_{x \to -1} f(x)$?
State clearly if the value of the limit does not exist.

The limit does not exist.

(d) A large spherical helium balloon is being inflated at a constant rate of 4800 $cm^3\ s^{-1}$.

At what rate is the radius of the balloon increasing when the volume of the balloon is $288\,000\pi$ cm^3?

You must use calculus and show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

v = volumn of balloon.

t = time.

r = radius of the balloon.

$V = \frac{4}{3}\pi r^3$.

$\frac{dv}{dr} = 4\pi r^2$. $\frac{dr}{dv} = \frac{1}{4\pi r^2}$.

$\frac{dv}{dt} = 4800$.

$\frac{dr}{dt} = \frac{dr}{dv} \cdot \frac{dv}{dt} = \frac{1200}{\pi r^2}$.

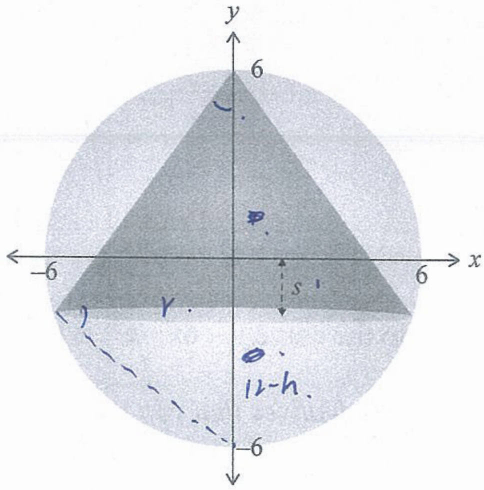
when $V = 288000\pi$. $\frac{4}{3}\pi r^3 = 288000\pi$.

$r^3 = 288000 \times \frac{3}{4}$.

$r = 60$

$\frac{dr}{dt} = \frac{1200}{\pi \times 3600} = \frac{1}{3\pi}$ $cm^3 s^{-1}$.

(e) A cone of height h and radius r is inscribed, as shown, inside a sphere of radius 6 cm.



The base of the cone is s cm below the x-axis.

$-(h^2-8h)$
$=-(h-4)^2.$

Find the value of s which maximises the volume of the cone.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the volume you have found is a maximum.

Cone : $V_c = \frac{1}{3}\pi r^2 h.$ $\Rightarrow$

$h = 6+s.$

$\frac{r}{12-h} = \frac{h}{r}.$

$r^2 = (12-h)h$

$r^2 = 12h-h^2.$

$\cancel{r^2 = 12(6+s)-(6+s)^2}$

$\Rightarrow.$

$V_c = \frac{1}{3}\pi.(12-h).h^2.$

$V_c = \frac{1}{3}\pi \cdot 12h^2 - \frac{1}{3}\pi \cdot h^3.$

$= 4\pi h^2 - \frac{1}{3}\pi h^3.$

$\frac{dV_c}{dh} = 8\pi h - \pi h^2.$

$\frac{dh}{ds} = 1.$

$\frac{dV_c}{ds} = 8\pi h - \pi h^2.$

$= \pi(-h^2+8h) = 0$

$-h^2+8h=0$

$\cancel{h^2-8h+16}$ $(h-4)^2=16$ $h = 8.$

$h=\pm4+4.$

$s = h-6 = 2\text{ cm}$

E8

QUESTION THREE

ASSESSOR'S USE ONLY

(a) Differentiate $f(x) = \sqrt[4]{3x+2}$.

$f'(x) = \cdot \ (3x+2)^{\frac{1}{4}}$

$= \frac{1}{4} \cdot 3 \cdot (3x+2)^{-\frac{3}{4}}$

$= \frac{3}{4} (3x+2)^{-\frac{3}{4}}$ //

(b) Find the x-value at which a tangent to the curve $y = 6x - e^{3x}$ is parallel to the x-axis.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = 6 - 3e^{3x}.$

$6 - 3e^{3x} = 0.$

$3e^{3x} = 6$

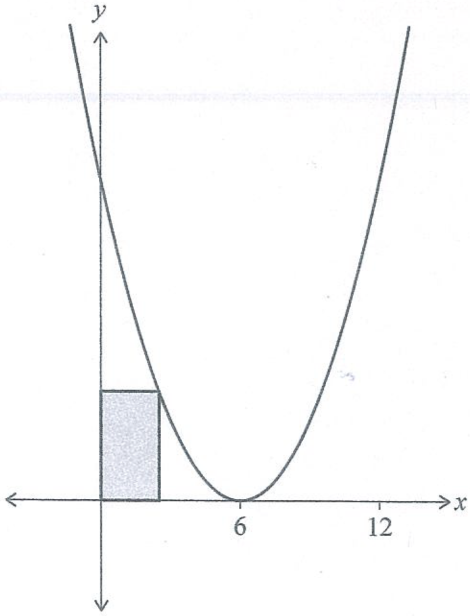
$e^{3x} = 2.$

$\ln \Rightarrow 3x = \ln 2.$

$x = 0.231$ //

(c) A rectangle has one vertex at (0,0) and the opposite vertex on the curve $y = (x-6)^2$, where $0 < x < 6$, as shown on the graph below.

ASSESSOR'S USE ONLY



Find the maximum possible area of the rectangle.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the area you have found is a maximum.

$A = \cdot x \cdot (x-6)^2.$

$f = x, \quad f' = 1,$

$g = (x-6)^2, \quad g' = 2(x-6).$

$\cancel{A} \quad \frac{dA}{dx} = x \cdot 2(x-6) + (x-6)^2.$

$= 2x(x-6) + (x^2 - 12x + 36)$

$= 2x^2 - 12x + x^2 - 12x + 36$

$= 3x^2 - 24x + 36$

$\cancel{= 3}$

$3x^2 - 24x + 36 = 0$

$x^2 - 8x + 12 = 0.$

$(x-4)^2 = 4$

$x = \pm 2 + 4.$

$\cancel{x = 6}$

x can't be. 6

so $x = 2.$

$y = 16.$

$A = x \cdot y = 32.$

ASSESSOR'S USE ONLY

(d) If $y = \dfrac{e^x}{\sin x}$, show that $\dfrac{dy}{dx} = y(1 - \cot x)$.

$y = \dfrac{e^x}{\sin x}$. $g = \sin x$, $g' = \cos x$.

$f = e^x$, $f' = e^x$.

$\cancel{\dfrac{dy}{dx}}$ $\dfrac{dy}{dx} = \dfrac{\sin x \cdot e^x - e^x \cdot \cancel{\sin x} \cdot \cos x}{\sin^2 x}$.

$= \cdot\ e^x \cdot \dfrac{\sin x - \cos x}{\sin^2 x}$.

$\cancel{= e^x \cdot \frac{1}{\sin x}}$

mei

$= e^x \cdot \dfrac{1}{\sin x} \cdot (1 - \cot \theta)$.

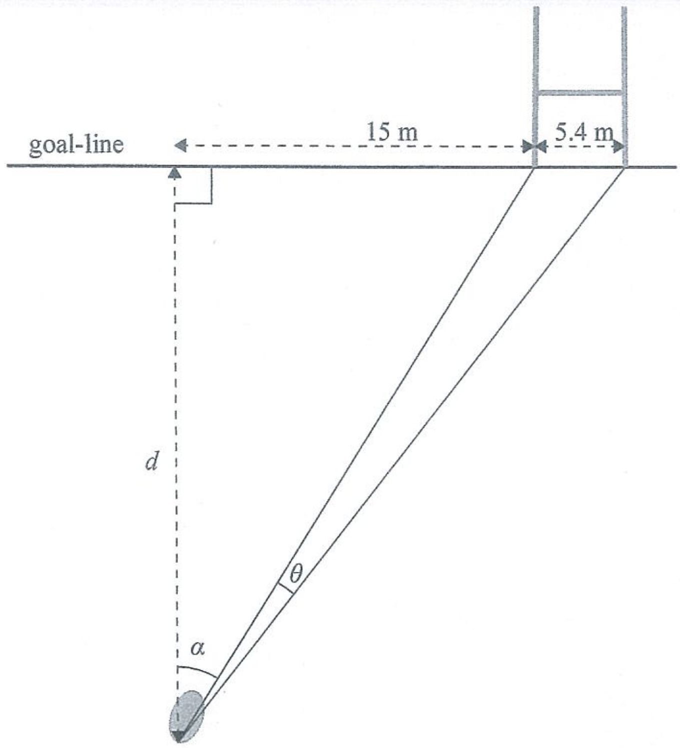
$= y \cdot (1 - \cot x)$

$=$ RHS

ASSESSOR'S USE ONLY

(e) In a rugby game, a try is scored 15 m from the left-hand goal-post. The conversion kick is taken at some point on the line perpendicular to the goal-line from the point where the try was scored, as shown in the diagram below.

The ball needs to pass between the goal-posts, which are 5.4 m apart.



Find the distance d from the goal-line that the conversion kick should be taken from in order to maximise the angle θ between the lines from the ball to the goal-posts.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the angle you have found is a maximum.

$\tan \alpha = \frac{15}{d}$.

$\tan(\alpha+\theta) = \frac{20.4}{d}$.

$\tan\theta = \tan(\alpha+\theta-\alpha) = \dfrac{\frac{20.4}{d} - \frac{15}{d}}{1+\frac{15\times 20.4}{d^2}} = \dfrac{\frac{5.4}{d}}{\frac{d^2+306}{d^2}} = \dfrac{5.4d}{d^2+306}$

$g = d^2+306,\ g' = 2d,\ f = 5.4d,\ f' = 5.4$

$\frac{d\tan\theta}{d\alpha} = \dfrac{(d^2+306)\times 5.4 - 2d\cdot 5.4d}{(d^2+306)^2} = \dfrac{1652.4 - 5.4d^2}{(d^2+306)^2}$

$\dfrac{1652.4 - 5.4d^2}{(d^2+306)^2} = 0$

$1652.4\ \cancel{1652.4} - 5.4d^2 = 0$

$d^2 = 306$

$d = 17.5$ m

E8

Excellence exemplar 2016

Subject:	Differentiation	Standard:	91578	Total score:	21

Q	Grade score	Annotation
1	M5	This question provides evidence towards M5 because the candidate successfully found the x-coordinate of the point P by applying their knowledge of derivatives and their understanding of the relationship between the gradients of two perpendicular lines. The candidate did not achieve M6 because after having found the correct gradient of -1 for the required tangent, they committed to a final answer of +1. When they attempted the excellence question (part e), they successfully used the chain rule to find the first derivative but did not realise that they needed to use the product rule to find the second derivative. Their final answer of $x = k$ was a very common response from the candidates and showed that not applying the product rule to find the second derivative was a common error.
2	E8	This candidate was able to construct an algebraic model for the volume of the cone in terms of h, find its first derivative and solve it equal to zero to find the h value that would result in the maximum volume of the cone. They could then use $h = 8$ to find the required s value that corresponded to the maximum volume of the cone. By completing the excellence question successfully they gained the maximum E8 for this question. However they did not successfully complete all the parts of question two. The candidate made a careless error in their working for part b where the power changed from negative ½ to positive ½ in their working to find the required derivative. Their response for part c showed gaps in their understanding about when the graph of a function is not differentiable, has a zero gradient, is concave down and is discontinuous but still has a limit.
3	E8	This candidate demonstrated a very high level of understanding of the topic by being able to use the compound angle formula for tan to construct an appropriate model for tan θ in terms of d. Their working showed but did not state explicitly that they realised that maximising tan θ would also maximise the angle θ between the lines from the ball to the goal posts since $y = \tan\theta$ is an increasing function over the interval from 0 to $\frac{\pi}{2}$. This candidate was able to differentiate the expression for $tan\theta$ in terms of d and solve it equal to zero to calculate the required distance of 17.5 m that would maximise the angle θ. This candidate completed all the parts of question three correctly.