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SUPERVISOR'S USE ONLY

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QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2016

91578 Apply differentiation methods in solving problems

9.30 a.m. Wednesday 23 November 2016
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–15 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Merit

TOTAL 15

ASSESSOR'S USE ONLY

QUESTION ONE

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(a) Differentiate $y = 1 + x - \frac{1}{x} + \frac{1}{x^2}$. $= 1 + x - x^{-1} + x^{-2}$

$y' = 1 + x^{-2} - 2x^{-3}$

(b) The height of the tide at a particular beach today is given by the function

$$h(t) = 0.8\sin\left(\frac{4\pi}{25}t + \frac{\pi}{2}\right)$$

$\sin(a x + b)$

$\frac{4\pi}{25}t + \frac{\pi}{2} = x$ $\quad 0.8\sin x$ $\quad 0.8\sin(\frac{4\pi}{25}t + \frac{\pi}{2})$ $\quad t = 0.8$ $\quad t' = 0$ $\quad y = \sin(\frac{4\pi}{25}t + \frac{\pi}{2})$ $\quad y' = \frac{4\pi}{25}\cos(\frac{4\pi}{25} + \frac{\pi}{2}$

where h is the height of water, in metres, relative to the mean sea level and t is the time in hours after midnight.

c2kiwi.blogspot.co.nz/2011/01/christchurch-wedding-stroll-on-beach.html

At what rate was the height of the tide changing at that beach at 9.00 a.m. today?

$h(t) = 0.8\sin(\frac{2\pi}{25}t + \frac{\pi}{2})$

$\frac{dh}{dt} = 0.8 \cdot \frac{4\pi}{25}\cos(\frac{4\pi}{25}t + \frac{\pi}{2})$

$= \frac{3.2\pi}{25}\cos(\frac{4\pi}{25}t + \frac{\pi}{2})$

$t = 9$

$\frac{dh}{dt} = \frac{3.2\pi}{25}\cos(\frac{4\pi}{25} \times 9 + \frac{\pi}{2})$

$= \frac{3.2\pi}{25}\cos(\frac{49\pi}{25} + \frac{\pi}{2}) = 0.395$

(c) A curve is defined by the parametric equations

$x = 2\cos 2t$ and $y = \tan^2 t$. $(\tan t)^2$

$y = (ax+b)^n$
$\frac{dy}{dx} = n(a)(ax+b)^{n-1}$ $y = ax^2$
$\frac{dy}{dx} = 2ax$

Level

Find the gradient of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

$\frac{dx}{dt} = -4\sin 2t$ $\frac{dy}{dt} = 2\sec^2 t \tan t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2\sec^2 t \tan t \cdot \frac{1}{-4\sin 2t}$

$= \frac{2\sec^2(t)\tan(t)}{-4\sin 2t}$

$= \frac{\sec^2 t \cdot \tan t}{-2\sin 2t}$

$t = \frac{\pi}{4}$

$= \frac{\sec^2\frac{\pi}{4} \cdot \tan\frac{\pi}{4}}{-2\sin(2\times\frac{\pi}{4})}$

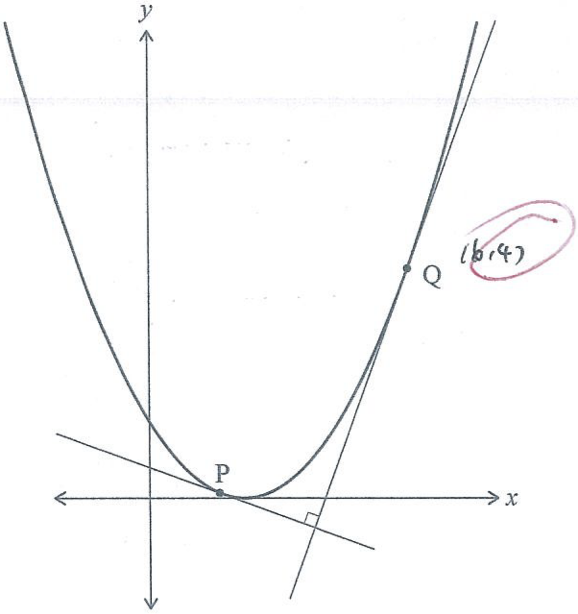
$= \frac{(\tan^2\frac{\pi}{4} + 1) \cdot \tan\frac{\pi}{4}}{-2\sin(2\times\frac{\pi}{4})}$

$= \frac{2 \cdot 1}{-2\sin\frac{\pi}{2}}$

$= \frac{2}{-2} = -1$

ASSESSOR'S USE ONLY

(d) The tangents to the curve $y = \frac{1}{4}(x-2)^2$ at points P and Q are perpendicular.



Q is the point $(6, 4)$.

What is the x-coordinate of point P?

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = \frac{1}{4}(x-2)^2$

$= \frac{1}{4}(x^2 + 4 - 4x)$

$= \frac{1}{4}x^2 + 1 - x$

~~$\frac{dy}{dx} = \frac{1}{2} \cdot 1 \cdot (x-2)$~~

~~$= \frac{1}{2}(x-2)$~~

~~$= \frac{1}{2}x - 1$~~

$\frac{dy}{dx} = \frac{1}{2}x - 1$

$x = 6$

$\frac{dy}{dx} = \frac{1}{2} \times 6 - 1 = 2$

$P: \frac{dy}{dx} = -\frac{1}{2}$

$-\frac{1}{2} = \frac{1}{2}x - 1$

$\frac{1}{2} = \frac{1}{2}x$

$x = 1$

$y = \frac{1}{4} \times (1-2)^2$

$= \frac{1}{4} \times 1$

$= \frac{1}{4}$

$P(1, \frac{1}{4})$

(e) A curve is defined by the function $f(x) = e^{-(x-k)^2}$.

ASSESSOR'S USE ONLY

Find, in terms of k, the x-coordinate(s) for which $f''(x) = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$-(x-k)^2$

$= -(x^2+k^2-2xk)$

$= -x^2-k^2+2kx$

$f = -2x+2k \quad f' = -2$

$g = e^{-(x-k)^2} \quad g' = (-2x+2k)e^{-(x-k)^2}$

$f'(x) = (-2x+2k)e^{-(x-k)^2}$

$f''(x) = 0$

$f''(x) = (-2x+2k)\cdot(-2x+2k)e^{-(x-k)^2} - 2e^{-(x-k)^2}$

$= (-2x+2k)^2 e^{-(x-k)^2} - 2e^{-(x-k)^2}$

$= e^{-(x-k)^2}\cdot[(-2x+2k)^2 - 2]$

$= e^{-(x-k)^2}\cdot(4x^2+4k^2-2) = 0$

$e^{-(x-k)^2} \neq 0$

$4x^2+4k^2-2 = 0$

$x^2+k^2 = \frac{1}{2}$

$x^2 = \frac{1}{2}-k^2$

$x = \pm\sqrt{\frac{1}{2}-k^2}$

M6

QUESTION TWO

ASSESSOR'S USE ONLY

(a) Differentiate $f(x) = x\ln(3x-1)$.

$f = x \quad f' = 1$

$g = \ln(3x-1) \quad g' = \frac{1}{3x-1}$

$f'(x) = x \cdot \frac{1}{3x-1} + \ln(3x-1)$

$= \frac{x}{3x-1} + \ln(3x-1)$

(b) Find the gradient of the tangent to the function $y = \sqrt{2x-1}$ at the point $(5,3)$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = (2x-1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} \cdot 2\,(2x-1)^{-\frac{1}{2}}$

$= (2x-1)^{-\frac{1}{2}}$

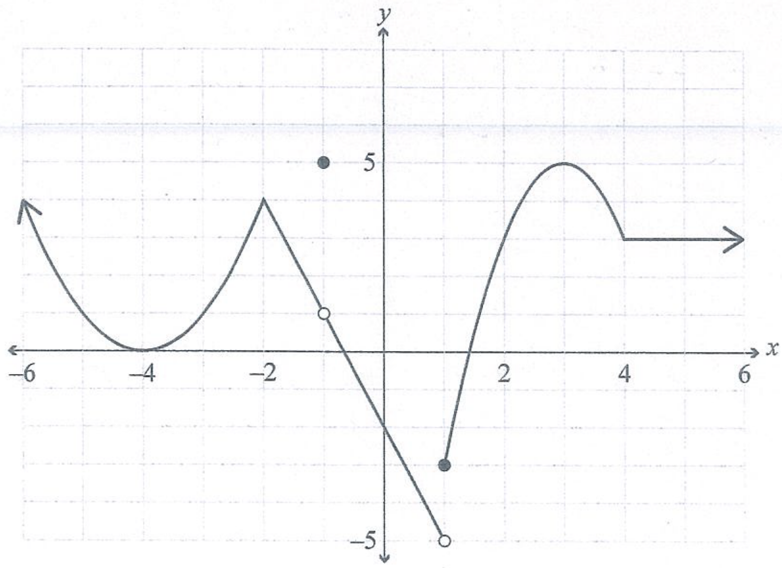
$x = 5$

$\frac{dy}{dx} = (2\times 5-1)^{-\frac{1}{2}}$

$= 9^{-\frac{1}{2}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

(c) The graph below shows the function $y = f(x)$.

ASSESSOR'S USE ONLY



For the function $y = f(x)$ above:

(i) Find the value(s) of x that meet the following conditions:

1. f is not continuous: 1
2. f is not differentiable: −2, 1, 4
3. $f'(x) = 0$: −4, 3
4. $f''(x) < 0$: $1 < x < 4$

(ii) What is the value of $\lim\limits_{x \to -1} f(x)$?

State clearly if the value of the limit does not exist.

1

(d) A large spherical helium balloon is being inflated at a constant rate of 4800 $cm^3 s^{-1}$.

At what rate is the radius of the balloon increasing when the volume of the balloon is $288\,000\pi$ cm^3?

You must use calculus and show any derivatives that you need to find when solving this problem.

ASSESSOR'S USE ONLY

$\frac{dv}{dt} = 4800$

$V = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dv} \cdot \frac{dv}{dt} = \frac{1}{4\pi r^2} \cdot 4800$

$= \frac{1200}{\pi r^2}$

$V = 288000\pi$

$288000\pi = \frac{4}{3}\pi r^3$

$288000 = \frac{4}{3} r^3$

$216000 = r^3$

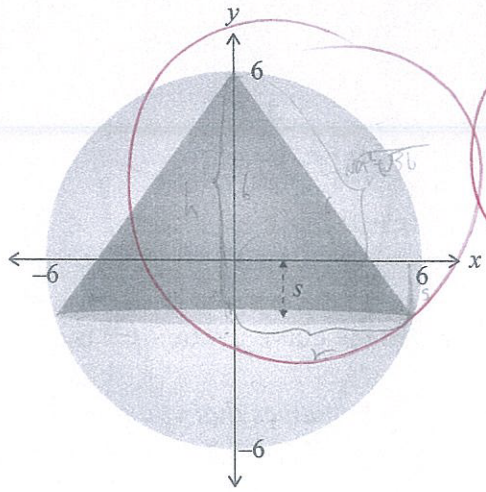
$r = 60$

$\frac{dr}{dt} = \frac{1200}{60^2\pi}$

$= 0.106 \text{ cm s}^{-1}$

(e) A cone of height h and radius r is inscribed, as shown, inside a sphere of radius 6 cm.

ASSESSOR'S USE ONLY



$h = 6+s$

$r = \cdots s$

$r = \cdots h$

The base of the cone is s cm below the x-axis.

Find the value of s which maximises the volume of the cone.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the volume you have found is a maximum.

$V = \frac{1}{3}\pi r^2 h \qquad h = 6+s$

$\cancel{\frac{dV}{dh} = \frac{2}{3}\pi r h}$

$\cancel{\tan\theta = \frac{r}{h}} \qquad \cancel{V = \frac{1}{3}\pi h \tan\theta\, h}$

$\cancel{r = h\tan\theta} \qquad \cancel{= \frac{1}{3}\pi h^2 \tan\theta}$

$\cancel{\tan\theta = \frac{6}{6}} \qquad V = \frac{1}{3}\pi r^2 (6+s)$

$= (2\pi + \frac{1}{3}\pi s) r^2$

M5

QUESTION THREE

ASSESSOR'S USE ONLY

(a) Differentiate $f(x) = \sqrt[4]{3x+2}$.

$f(x) = (3x+2)^{\frac{1}{4}}$

$f'(x) = \frac{1}{4} \cdot 3 \cdot (3x+2)^{-\frac{3}{4}}$

$= \frac{3}{4}(3x+2)^{-\frac{3}{4}}$

(b) Find the x-value at which a tangent to the curve $y = 6x - e^{3x}$ is parallel to the x-axis.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = 6 - 3e^{3x}$

$\frac{dy}{dx} = 0$

$6 - 3e^{3x} = 0$

$6 = 3e^{3x}$

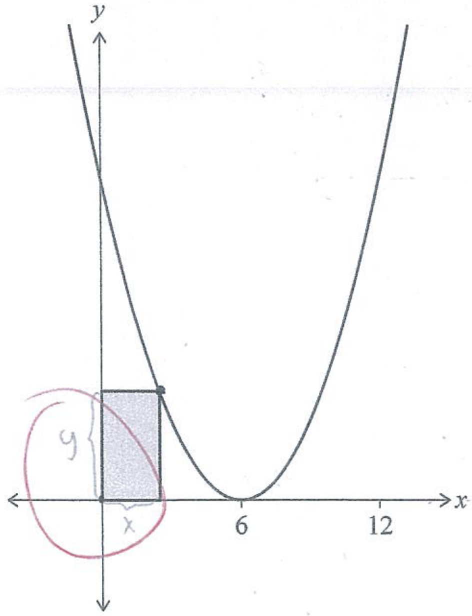
$2 = e^{3x}$

$\ln 2 = 3x$

$x = \frac{1}{3}\ln 2 = 0.231$

(c) A rectangle has one vertex at (0,0) and the opposite vertex on the curve $y = (x - 6)^2$, where $0 < x < 6$, as shown on the graph below.

ASSESSOR'S USE ONLY



Find the maximum possible area of the rectangle.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the area you have found is a maximum.

$$A = xy = x(x-6)^2$$
$$= x(x^2+36-12x)$$
$$= x^3+36x-12x^2$$
$$A' = 3x^2+36-24x$$
$$0 = 3x^2+36-24x$$
$$0 = x^2+12-8x$$
$$x^2-8x+12=0$$
$$(x-2)(x-6)=0$$
$$x=2 \quad x=6$$
$$0<x<6$$
$$\therefore x=2$$

A

ASSESSOR'S USE ONLY

(d) If $y = \dfrac{e^x}{\sin x}$, show that $\dfrac{dy}{dx} = y(1 - \cot x)$.

$f = e^x \quad f' = e^x$

$g = \sin x \quad g' = \cos x$

$$\frac{dy}{dx} = \frac{\sin x \cdot e^x - e^x \cdot \cos x}{\sin^2 x}$$

$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}$$

$$= y(\sin x - \cos x)$$

$$= y(1 - \cot x)$$

(e) In a rugby game, a try is scored 15 m from the left-hand goal-post. The conversion kick is taken at some point on the line perpendicular to the goal-line from the point where the try was scored, as shown in the diagram below.

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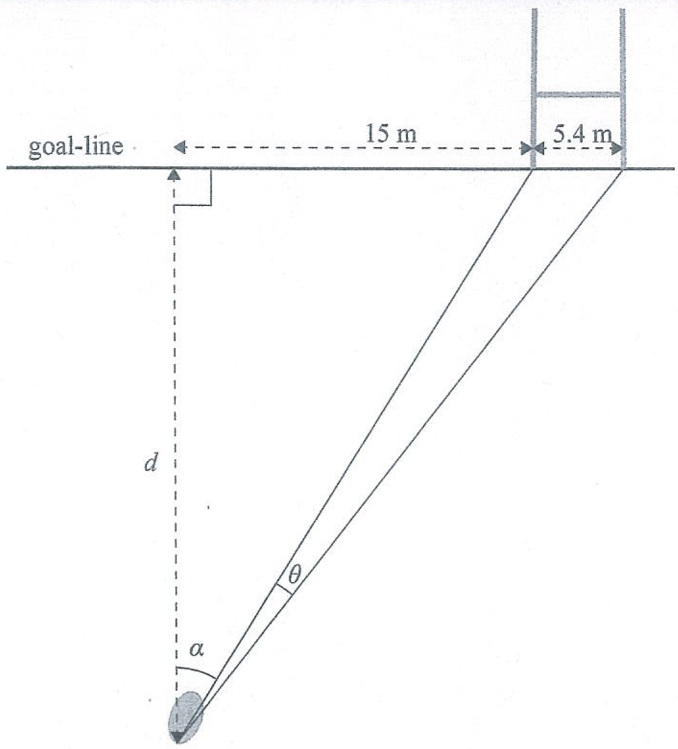
The ball needs to pass between the goal-posts, which are 5.4 m apart.



Find the distance d from the goal-line that the conversion kick should be taken from in order to maximise the angle θ between the lines from the ball to the goal-posts.

You must use calculus and show any derivatives that you need to find when solving this problem.

You do not need to prove that the angle you have found is a maximum.

$$\tan(a+\theta) = \frac{15+5.4}{d}$$

$$\tan a = \frac{15}{d}$$

$$\tan(a+\theta) = \frac{20.4}{d}$$

$$\cancel{20.4 = d\tan(a+\theta)}$$

$$\tan a = \frac{15}{d}$$

A4

Merit exemplar 2016

Subject:	Differentiation	Standard:	91578	Total score:	15

Q	Grade score	Annotation
1	M6	This question provides evidence for an M6 because the candidate has successfully completed two of the r questions. In the excellence question they have found the second derivative of the exponential function but have not been able to solve the second derivative equal to zero correctly. This candidate has also shown that they can find and use the gradient function for the curve defined by the parametric equations involving cos and tan in part c to find the required gradient. They actually achieved three r codes in this question because they also found the required x-coordinate for the point of contact of the tangent at the point P.
2	M5	This question provides evidence for an M5 because the candidate was able to solve the related rates of change problem involving the volume of spherical balloon. This candidate did not achieve higher than the M5 code for this question because they did not find all the required x values for the different specified features of the graph in part c and were not able to set up the model required to solve the volume of the cone problem in part e.
3	A4	This question provides evidence for an A4 because both of the achievement level questions parts a and b have been successfully completed and both of the merit questions have been partially completed. For part c, this candidate has constructed the correct model for the area of the rectangle, differentiated it and solved it equal to zero. They have then made the correct choice for the x value that would result in the maximum possible area. They have stopped there however without answering the question as to what the maximum possible area would be that would correspond to the x value of 2 already found. For part d, they have used the quotient rule correctly to find the derivative of the given function. They have made an error in their second to last line of working as they tried to show that their derivative could be rewritten in the alternative form provided. No progress was achieved in their efforts to solve the excellence part of this question.