91578

3

SUPERVISOR'S USE ONLY

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

# Level 3 Calculus, 2017

## 91578 Apply differentiation methods in solving problems

9.30 a.m. Thursday 23 November 2017
Credits: Six

| Achievement | Achievement with Merit | Achievement with Excellence |
|---|---|---|
| Apply differentiation methods in solving problems. | Apply differentiation methods, using relational thinking, in solving problems. | Apply differentiation methods, using extended abstract thinking, in solving problems. |

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

**You should attempt ALL the questions in this booklet.**

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

**YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.**

Merit

TOTAL 17

ASSESSOR'S USE ONLY

ASSESSOR'S USE ONLY

## QUESTION ONE

(a) Differentiate $y = \sqrt{x} + \tan(2x)$.

$y = x^{\frac{1}{2}} + \tan 2x$

$y' = \frac{1}{2}x^{-\frac{1}{2}} + 2\sec^2 x$

(b) Find the gradient of the tangent to the curve $y = \frac{e^{2x}}{x+2}$ at the point where $x = 0$.

*You must use calculus and show any derivatives that you need to find when solving this problem.*

$f = e^{2x}$ $\quad f' = 2e^{2x}$

$g = x + 2$ $\quad g' = 1$

$y' = \frac{(x+2)2e^{2x} - e^{2x}}{(x+2)^2}$ ~~to~~

$\cancel{(x+2)2e^{2x} - e^{2x} = 0}$

$\cancel{e^{2x}((2x+4) - 1) = 0}$

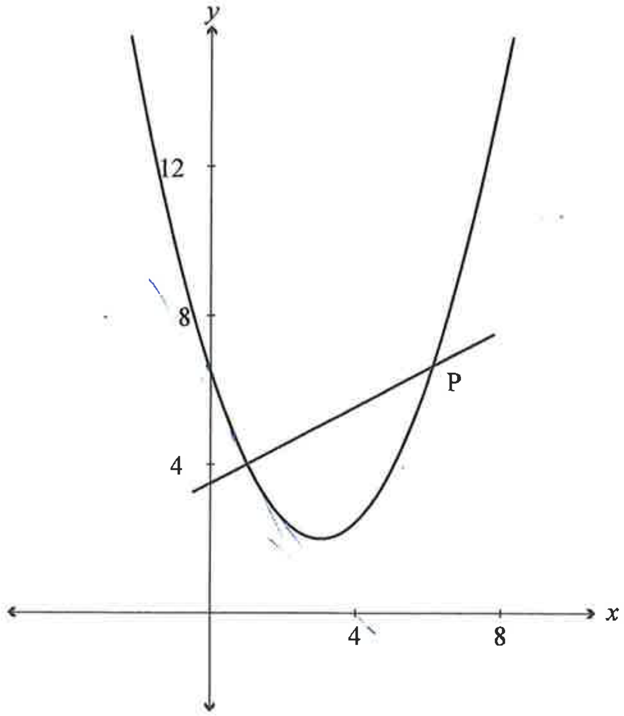
$\cancel{2x + 4 - 1 = 0}$

$y' = \frac{(0+2)(2e^{2(0)}) - e^{2(0)}}{(0+2)^2}$

$= 0.75$

(c) The normal to the parabola $y = 0.5(x-3)^2 + 2$ at the point (1,4) intersects the parabola again at the point P.

ASSESSOR'S USE ONLY




Find the $x$-coordinate of point P.

*You must use calculus and show any derivatives that you need to find when solving this problem.*

$y' = 2 \times 0.5(x-3) = x-3$

$y' = 1-3 = -2$ grad. normal $= \frac{1}{2}$

$y - 4 = \frac{1}{2}(x-1)$

$y - 4 = \frac{1}{2}x - \frac{1}{2}$

$y = \frac{1}{2}x + \frac{7}{2}$

$\frac{1}{2}x + \frac{7}{2} = 0.5(x-3)^2 + 2$

$\frac{1}{2}x + \frac{3}{2} = 0.5x^2 - 3x + 4.5$

$0 = 0.5x^2 - \frac{7}{2}x + 6$

$0 = (x-4)(x-3)$

$x = 4 \quad x = 3$

$P = 4$

(d) A curve is defined parametrically by the equations $x = \sqrt{t+1}$ and $y = \sin 2t$.

Find the gradient of the tangent to the curve at the point when $t = 0$.

*You must use calculus and show any derivatives that you need to find when solving this problem.*

$x = (t+1)^{\frac{1}{2}}$ $\qquad$ $y = \sin 2t$

$\frac{dx}{dt} = \frac{1}{2\sqrt{t+1}}$ $\qquad$ $\frac{dy}{dt} = 2\cos 2t$

$\frac{dy}{dx} = 2\cos 2t \times 2\sqrt{t+1}$

$\frac{dy}{dx} = 2\cos 2(0) \times 2\sqrt{(0)+1} = 4$

ASSESSOR'S USE ONLY

(e) Find the values of $a$ and $b$ such that the curve $y = \dfrac{ax-b}{x^2-1}$ has a turning point at (3,1).

*You must use calculus and show any derivatives that you need to find when solving this problem.*

ASSESSOR'S USE ONLY

$y \quad f = ax - b \qquad f' = a$

$g = x^2 - 1 \qquad g' = 2x$

$y' = \dfrac{(ax^2 - a) - 2x(ax - b)}{(x^2-1)^2}$

$0 = \dfrac{ax^2 - a - 2ax^2 + 2bx}{(x^2-1)^2}$

$0 = \dfrac{a(3)^2 - a - 2a(3)^2 + 2b(3)}{(3^2-1)^2}$

$0 = 9a - a - 18a + 6b$

$0 = -10a + 6b$

$10a = 6b \qquad a = 0.6B$

$1 = \cancel{(0.6B)(3) - B} \qquad \dfrac{0.6B(3) - B}{3^2 - 1}$

$8 = 0.6B^2 - B \qquad B =$

M6

ASSESSOR'S USE ONLY

## QUESTION TWO

(a) Differentiate $y = 2(x^2 - 4x)^5$.

*You do not need to simplify your answer.*

$f = 2 \quad f' = 0 \qquad g = (x^2 - 4x)^5 \qquad g' = 5(x^2-4x)^4(2x-4)$

$y' = 10(x^2 - 4x)^4(2x - 4)$

(b) The percentage of seeds germinating depends on the amount of water applied to the seedbed that the seeds are sown in, and may be modelled by the function:

$$P(w) = 96\ln(w + 1.25) - 16w - 12$$

where $P$ is the percentage of seeds that germinate and
$w$ is the daily amount of water applied (litres per square metre of seedbed), with $0 \leq w \leq 15$.

Find the amount of water that should be applied daily to maximise the percentage of seeds germinating.

*You must use calculus and show any derivatives that you need to find when solving this problem.*

$P'(w) = \dfrac{96}{w+1.25} - 16 = 0$

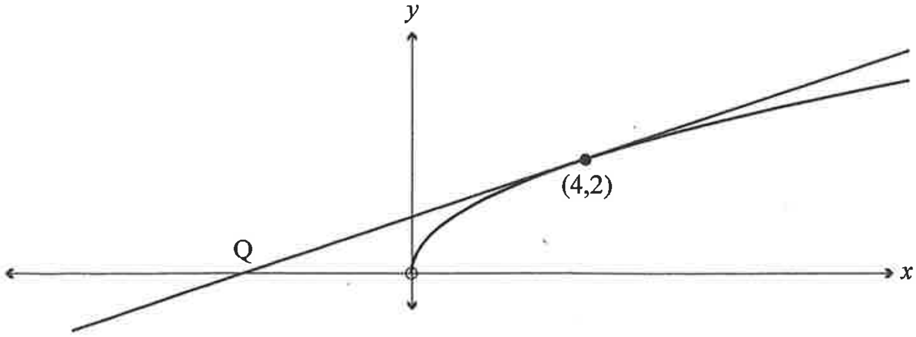
$\dfrac{96}{w+1.25} = 16$

$96 = 16(w + 1.25) = 16w + 20$

$16w = 76$

$w = 4.75$

(c) The tangent to the curve $y = \sqrt{x}$ is drawn at the point (4,2).




Find the co-ordinates of the point Q where the tangent intersects the $x$-axis.

*You must use calculus and show any derivatives that you need to find when solving this problem.*

$y = x^{\frac{1}{2}}$ $\quad$ $y' = \frac{1}{2}x^{-\frac{1}{2}}$

$y' = \frac{1}{2} \cancel{x} (4)^{-\frac{1}{2}} = 0.25$

$\cancel{y' = \frac{1}{2}(16)^{-\frac{1}{2}} = }$

$y - 2 = 0.25(x - 4)$ $\quad$ $y - 2 = 0.25x - 1$

$y = 0.25x + 1$

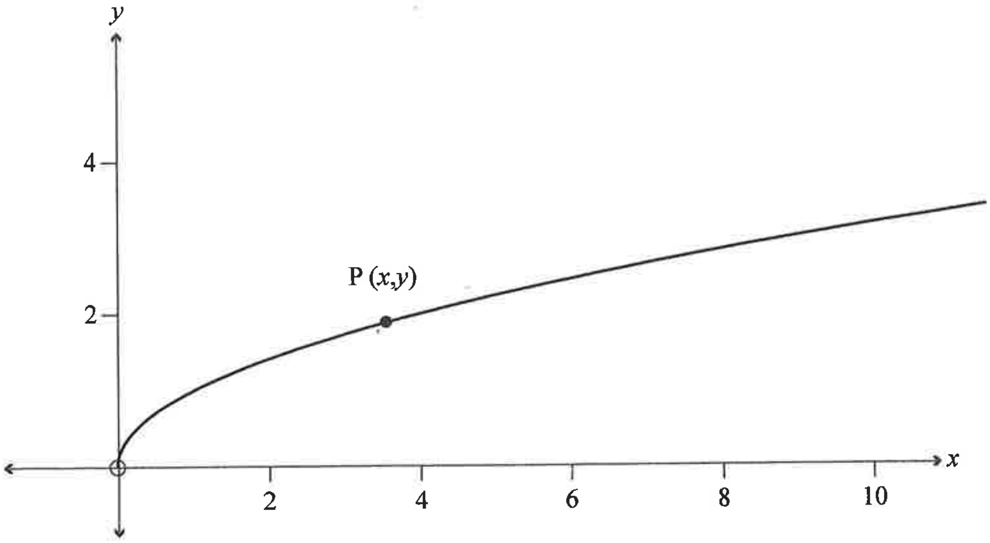
$\cancel{y = 0.25(0) + 1}$ $\quad$ $0 = 0.25x + 1$

$-1 = 0.25x$ $\quad$ $x = -4$

co-ordinates = $(-4, 0)$.

(d) Find the coordinates of the point P $(x,y)$ on the curve $y=\sqrt{x}$ that is closest to the point (4,0).






*You do not need to prove that your solution is the minimum value.*

*You must use calculus and show any derivatives that you need to find when solving this problem.*

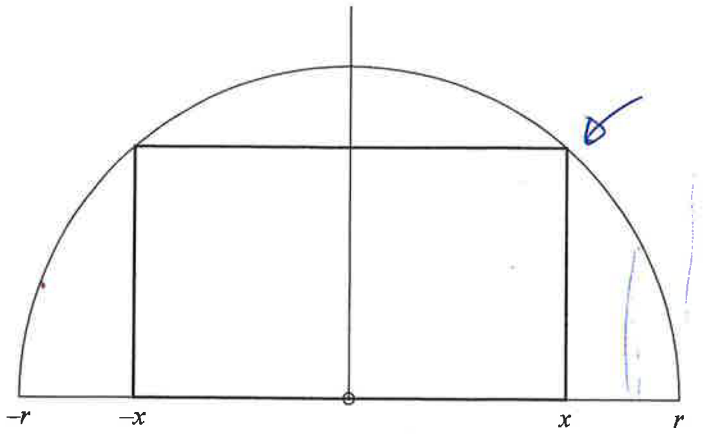
$\cancel{y' = \frac{1}{2}x^{-\frac{1}{2}}}$

$y = \sqrt{4}$ $\quad y = 2.$

$P = (4, 2).$

(e) A rectangle is inscribed in a semi-circle of radius $r$, as shown below.

ASSESSOR'S USE ONLY




Show that the maximum possible area of such a rectangle occurs when $x = \frac{r}{\sqrt{2}}$.

*You do not need to prove that your solution gives the maximum area.*

*You must use calculus and show any derivatives that you need to find when solving this problem.*

$\cancel{A = 2\left(\frac{r}{\sqrt{2}}\right)^2 \quad A = \frac{2r^2}{2} \quad A = r^2}$

$A = 2\left(\frac{1}{4} \times \pi r^2 - x^2\right)$

$A = \frac{\pi r^2}{2} - 2x^2$

$A' = \pi r - 4x = 0$

$\pi r = 4x$

$x = \frac{\pi r}{4}$

$x = \frac{r}{\sqrt{2}}$

M5

**QUESTION THREE**

(a) Differentiate $y = x\ln(3x-1)$.

*You do not need to simplify your answer.*

$f = x \quad f' = 1 \quad g = \ln(3x-1) \quad g' = \frac{3}{3x-1}$

$y' = \frac{3x}{3x-1} + \cancel{g}\ln(3x-1)$

(b) Find the gradient of the curve $y = \frac{1}{x} - \frac{1}{x^2}$ at the point $\left(2, \frac{1}{4}\right)$.

*You must use calculus and show any derivatives that you need to find when solving this problem.*

$y = x^{-1} - x^{-2}$

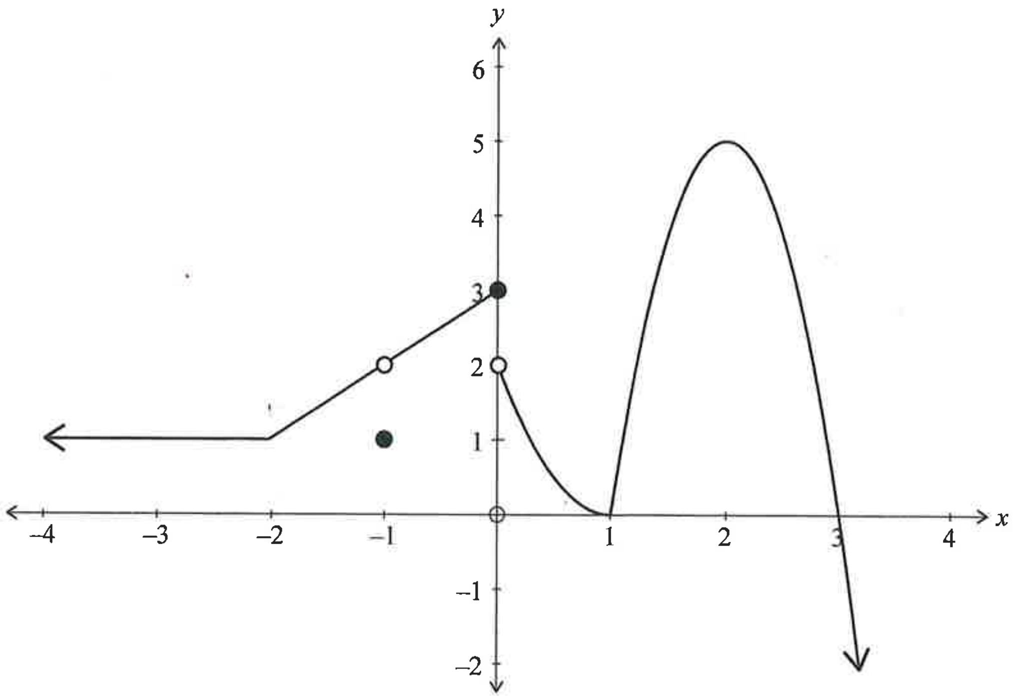
$y' = \frac{-x}{x^2} + \frac{2}{x^3}$

$y' = \frac{-2}{2^2} + \frac{2}{2^3}$

$= -\frac{1}{4}$

ASSESSOR'S USE ONLY

(c) The graph below shows the function $y = f(x)$.




For the function above:

(i) Find the value(s) of $x$ that meet the following conditions:

(1) $f'(x) = 0$: $x = 2$ ,

(2) $f(x)$ is continuous but not differentiable: $x = -2$ , $x = 1$

(3) $f(x)$ is not continuous: $x = -1$ , $x = 0$

(4) $f''(x) < 0$: $1 < x$

(ii) What is the value of $\lim\limits_{x \to -1} f(x)$? 2

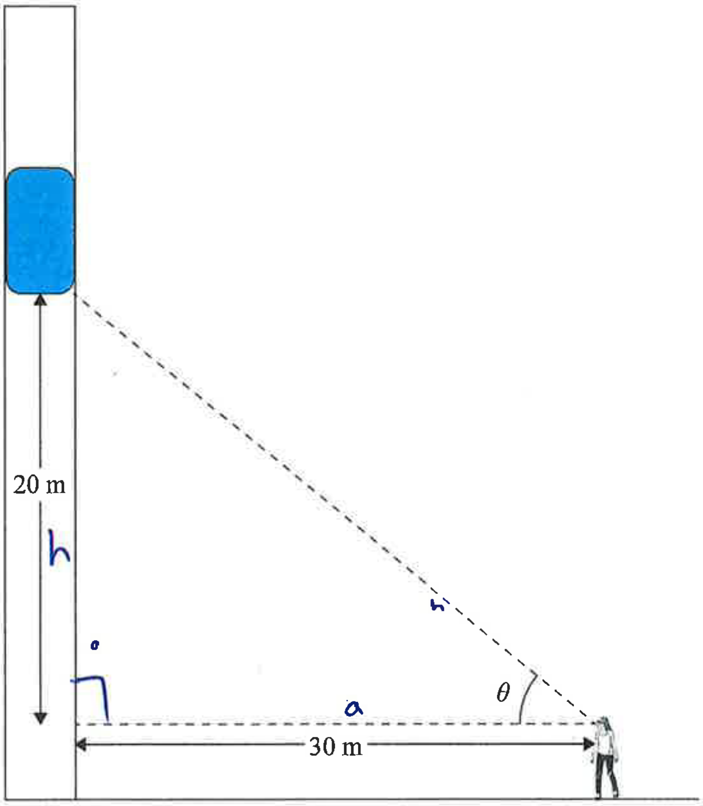
State clearly if the value does not exist.

(d) A building has an external elevator. The elevator is rising at a constant rate of 2 m $s^{-1}$.

Sarah is stationary, watching the elevator from a point 30 m away from the base of the elevator shaft.

Let the angle of elevation of the elevator floor from Sarah's eye level be $\theta$.

www.alibaba.com/product-detail/Sicher-external-elevator_60136882005.html




Find the rate at which the angle of elevation is increasing when the elevator floor is 20 m above Sarah's eye level.

*You must use calculus and show any derivatives that you need to find when solving this problem.*

$$\frac{d\theta}{dt} = \frac{dh}{dt} \times \frac{d\theta}{dh} \qquad \frac{dh}{dt} = 2.$$

$$\theta = \tan\left(\frac{h}{30}\right)$$

$$\frac{d\theta}{dh} = \frac{1}{30}\sec^2\left(\frac{h}{30}\right)$$

$$\frac{d\theta}{dt} = 2 \times \frac{\sec^2\left(\frac{h}{30}\right)}{30} \qquad h = 20$$

$$\frac{d\theta}{dt} = 0.07^\circ$$

(e) For the function $y = e^x \cos kx$:

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$f = e^x \quad f' = e^x \quad g = \cos kx \quad g' = -k \cancel{\cos}\ \sin kx$

$\frac{dy}{dx} = e^x(-k\sin kx) + e^x \cos kx$

$f = e^x \quad f' = e^x \quad g = -k\sin kx \quad g' = -k^2\cos kx$

$f = e^x \quad f' = e^x \quad g = \cos kx \quad g' = -k\sin kx$

$\frac{d^2y}{dx^2} = -k\sin kx e^x - k^2\cos kx e^x + \cos kx e^x - k\sin kx e^x$

$= \cancel{-2k\sin kx . e^x + \cos kx e^x(-k^2+k)}$

$= -2k\sin kx e^x - k^2\cos kx e^x + \cos kx e^x$

(ii) Find all the value(s) of $k$ such that the function $y = e^x \cos kx$ satisfies the equation

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ for all values of $x$.

$- 2k\sin kx e^x - k^2\cos kx e^x$

$-2k\sin kx . e^x \cancel{+2\cos kx\ e^x(-k^2+1)}) - 2(-k\sin kx . e^x + \cos kx e^x)$

$+ e^x \cos kx = 0$

$= -4k\sin kx . e^x - k^2\cos kx e^x = 0$

$-4k\sin kx . e^x = k^2\cos kx . e^x$

$-4k = k^2$

$k^2 + 4k = 0$

$k(k+4) = 0$

$k = 0 \quad k = -4$

ASSESSOR'S USE ONLY

M6

## Merit exemplar

| Subject: | Level 3 Calculus | Standard: | 91578 | Total score: | 17 |
|---|---|---|---|---|---|

| Q | Grade score | Annotation |
|---|---|---|
| 1 | M6 | This question provides evidence towards M6 because the candidate has correctly completed part 1d by using parametric differentiation to find $\frac{dy}{dx}$ and to then accurately substitute $t = 0$ to evaluate the required gradient of 4.<br>The candidate has also partially completed the excellence problem, part 1e. They have successfully used the quotient rule to find $\frac{dy}{dx}$ and have substituted $x = 3 \text{ and } \frac{dy}{dx} = 0$ to rearranged the resulting equation to find a correct relationship between the pronumerals, a and b for which they gain an r. They would have needed to substitute the given point (3, 1) into the original function to find a second relationship between a and b and then solved the resulting the simultaneous equations if they were to gain the E8 for this question. |
| 2 | M5 | The candidate provides evidence for M5 by correctly completing part 2c. In this question they have demonstrated that they are able to use calculus to find the equation of the tangent and then its x-axis intercept by substituting $y = 0$ into the equation of the tangent.<br>The candidate was not able to gain an M6 because they could not form the required model in part 2d for the distance between a point on the square root function and the point (4, 0). Similarly in the part 2e, this candidate was not able to form an appropriate model for the rectangle inscribed in the semi-circle provided. |
| 3 | M6 | The candidate gained an r code when they correctly identified the appropriate x values for three of the four features required in part 3c(i) as well as clearly stating the limit required in part 3c(ii).<br>They gained a M6 rather than an M5 because they gained a second r code when they demonstrated that they were able to find the second derivative of the function $y = e^x coskx$ by applying the product rule and chain rules successfully in part 3e(i).<br>They were not able to achieve an E7 or an E8 because they were not successful in their attempt to substitute into the differential equation of part 3e(ii) and therefore did not find the correct equation needing to be solved for this problem. |