3

SUPERVISOR'S USE ONLY

91578

915780

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2017

91578 Apply differentiation methods in solving problems

9.30 a.m. Thursday 23 November 2017
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 20

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QUESTION ONE

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(a) Differentiate $y = \sqrt{x} + \tan(2x)$.

$y = (x)^{\frac{1}{2}} + \tan(2x)$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \sec^2(2x) \times 2$

$= \frac{1}{2\sqrt{x}} + 2\sec^2(2x)$

(b) Find the gradient of the tangent to the curve $y = \frac{e^{2x}}{x+2}$ at the point where $x = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \frac{2e^{2x}(x+2) - e^{2x}(1)}{(x+2)^2}$

when $x=0$,

$\frac{dy}{dx} = \frac{2e^{2\times 0}(0+2) - e^{2\times 0}\times(1)}{(0+2)^2}$

$= \frac{2\times 2 - 1\times 1}{4}$

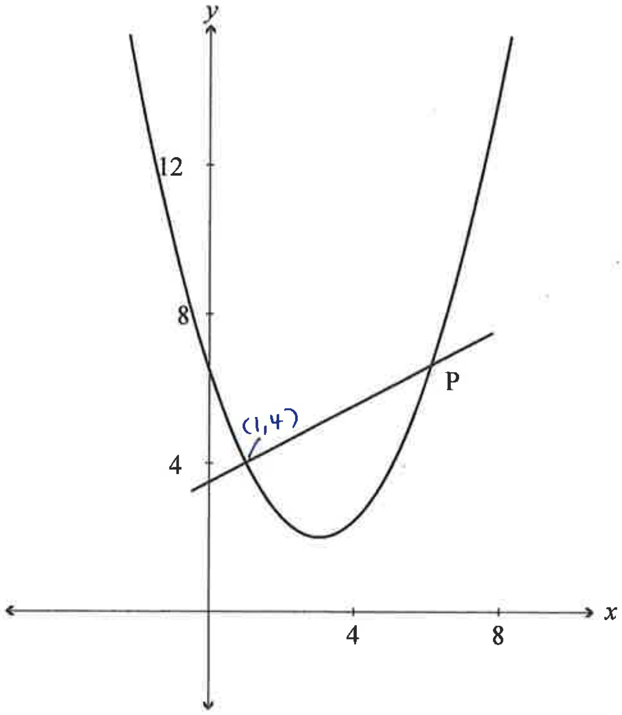
$= \frac{4-1}{4}$

$= \frac{3}{4}$

$\therefore \frac{dy}{dx} = \frac{3}{4}$

ASSESSOR'S USE ONLY

(c) The normal to the parabola $y = 0.5(x-3)^2 + 2$ at the point (1,4) intersects the parabola again at the point P.



Find the x-coordinate of point P.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \cancel{1}(x-3) \times 1$

$= x - 3$

At (1,4), the gradient of normal:

$\frac{dy}{dx} = 1 - 3 = -2$.

$((-2)\times(-1))^{-1} = \frac{1}{2}$

Equation of normal line $= y_{normal} = \frac{1}{2}x + C$.

$4 = \frac{1}{2}\times 1 + c$

$C = \frac{7}{2}$

$\therefore y_{normal} = \frac{1}{2}x + \frac{7}{2}$

Finding Intercepts : $y = 0.5(x-3)^2 + 2 = y_{normal} = \frac{1}{2}x + \frac{7}{2}$

$0.5(x-3)^2 + 2 = \frac{1}{2}x + \frac{7}{2}$

$0.5(x^2 - 6x + 9) + 2 = \frac{1}{2}x + \frac{7}{2}$

$0.5x^2 - 3x + 4.5 + 2 = \frac{1}{2}x + \frac{7}{2}$

$0.5x^2 - \frac{7}{2}x + 3 = 0$

$x = 1$ or $x = 6$

$\therefore x$-coordinate of point P $= 6$

ASSESSOR'S USE ONLY

(d) A curve is defined parametrically by the equations $x = \sqrt{t+1}$ and $y = \sin 2t$.

$(t+1)^{\frac{1}{2}}$

Find the gradient of the tangent to the curve at the point when $t = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dt} = \cos 2t \times 2 = 2\cos 2t$$

$$\frac{dx}{dt} = \frac{1}{2}(t+1)^{-\frac{1}{2}} = \frac{1}{2} \times \frac{1}{\sqrt{t+1}} = \frac{1}{2\sqrt{t+1}}$$

$$= 2\cos 2t \times \frac{2\sqrt{t+1}}{1}$$

$$= 4\cos 2t \times \sqrt{t+1}$$

When $t=0$, $\frac{dy}{dx} = 4\cos(2\times 0) \times \sqrt{0+1}$

$$= 4\cos(0) \times 1$$

$$= 4$$

$\frac{dy}{dx} = 4$, when $t=0$.

(e) Find the values of a and b such that the curve $y = \frac{ax - b}{x^2 - 1}$ has a turning point at (3,1).

min or max $f'(x) = 0$

You must use calculus and show any derivatives that you need to find when solving this problem.

$$\frac{dy}{dx} = \frac{a(x^2-1) - (ax-b)\times(2x)}{(x^2-1)^2}$$

when at turning point, $\frac{dy}{dx} = 0$. and $x = 3$.

$$\frac{a(x^2-1) - (ax-b)\times(2x)}{(x^2-1)^2} = 0$$

$$a(x^2-1) - 2x(ax-b) = 0$$

$$ax^2 - a - 2ax^2 + 2bx = 0$$

$$-ax^2 - a + 2bx = 0.$$

(substitute $x=3$) $-a(3)^2 - a + 2b(3) = 0.$

$$-9a - a + 6b = 0$$

① … $-10a + 6b = 0.$

$y = \frac{ax-b}{x^2-1}$ substitute (3,1).

$1 = \frac{a\times 3 - b}{3^2 - 1}$

$1 = \frac{3a-b}{8}$

② … $8 = 3a - b$

Simultaneous ① and ②

① … $-10a + 6b = 0. \xrightarrow{\div 2} -5a + 3b = 0.$

② … $3a - b = 8 \xrightarrow{\times 3} 9a - 3b = 24$

$+$

$4a = 24$

$a = 6$

$3a - b = 8$

$3\times 6 - b = 8$

$-b = -10$

$b = 10$

$\therefore a = 6$

$b = 10$

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E8

ASSESSOR'S USE ONLY

QUESTION TWO

(a) Differentiate $y = 2(x^2 - 4x)^5$.

You do not need to simplify your answer.

$\frac{dy}{dx} = 10(x^2-4x)^4 \times (2x-4)$

(b) The percentage of seeds germinating depends on the amount of water applied to the seedbed that the seeds are sown in, and may be modelled by the function:

$$P(w) = 96\ln(w + 1.25) - 16w - 12$$

where P is the percentage of seeds that germinate and
w is the daily amount of water applied (litres per square metre of seedbed), with $0 \leq w \leq 15$.

Find the amount of water that should be applied daily to maximise the percentage of seeds germinating.

P. $\frac{dP}{dw} = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dP}{dw} = 96 \times \frac{1}{w+1.25} - 16$

When P is maximum, $\frac{dP}{dw} = 0$

$\frac{96}{w+1.25} - 16 = 0$

$\frac{96}{w+1.25} = 16$

$\frac{6}{w+1.25} = 1$

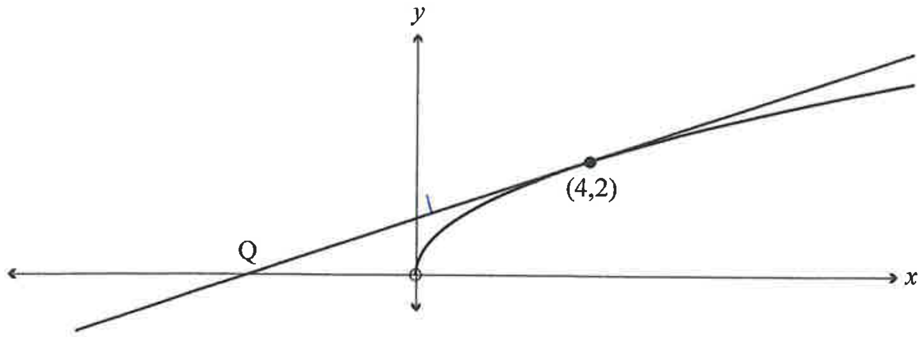
$6 = w + 1.25$

$w = 4.75$

$\therefore w = 4.75$ L per m^2

$(x)^{\frac{1}{2}}$

(c) The tangent to the curve $y = \sqrt{x}$ is drawn at the point (4,2).



Find the co-ordinates of the point Q where the tangent intersects the x-axis. when $y=0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

when $x=4$, $\frac{dy}{dx} = \frac{1}{2} \times (4)^{-\frac{1}{2}}$

$= \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{4}(x-4)$

$y - 2 = \frac{1}{4}x - 1$

$y = \frac{1}{4}x + 1$

Substitute $y=0$ at point Q.

$0 = \frac{1}{4}x + 1$

$-1 = \frac{1}{4}x$

$x = -1 \times 4$

$= -4$

$\therefore$ Coordinates of Q : $(-4, 0)$

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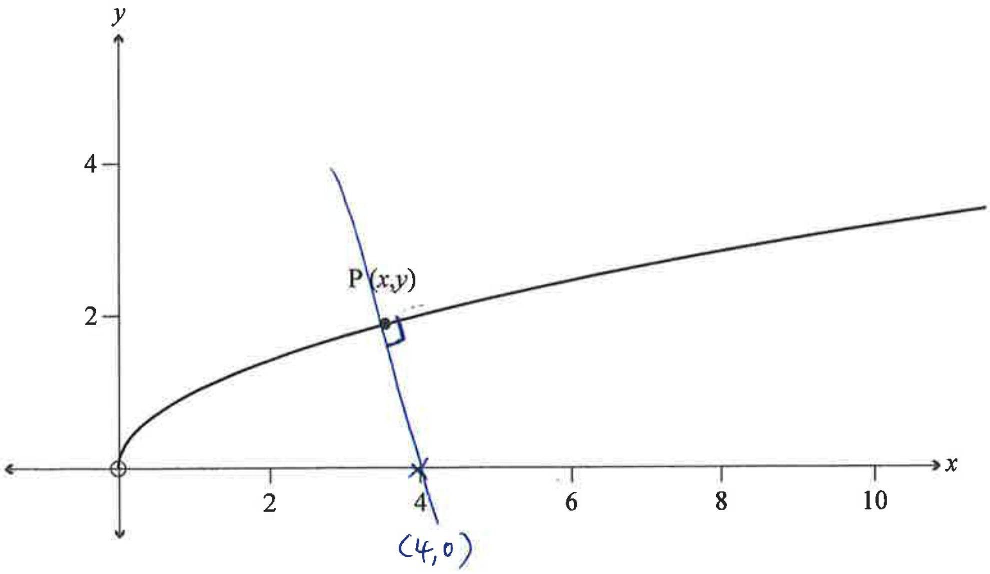
ASSESSOR'S USE ONLY

(d) Find the coordinates of the point P (x,y) on the curve $y=\sqrt{x}$ that is closest to the point (4,0).



You do not need to prove that your solution is the minimum value.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

To be closest to the point (4,0), the ~~normal line at~~ P(x,y) should have the normal line (perpendicular to the tangent line) which includes the point (4,0)

gradient of Normal line : $\left(\frac{1}{2}x^{-\frac{1}{2}} \times (-1)\right)^{-1}$

$= \left(-\frac{1}{2\sqrt{x}}\right)^{-1}$

$= -2\sqrt{x}$.

$y - y_1 = m(x - x_1)$

$y - 0 = -2\sqrt{x}(x-4)$

$y = -2x\sqrt{x} + 8\sqrt{x}$.

P is an intersept between $y=\sqrt{x}$ and $y=-2x\sqrt{x}+8\sqrt{x}$.

$\sqrt{x} = -2x\sqrt{x} + 8\sqrt{x}$

$\cancel{x} = -2x^{\cancel{2}} + 8\cancel{x}$

$1 = -2x+8$

$-7 = -2x$

$x = \frac{7}{2}$

$x = \frac{7}{2}$ and $y = 1.87$
$\left(\sqrt{\frac{7}{2}}\right)$

$y = \sqrt{x}$

$= \sqrt{\frac{7}{2}} = 1.87$

$\therefore P\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$

or

$P\left(\frac{7}{2}, 1.87\right)$

(e) A rectangle is inscribed in a semi-circle of radius r, as shown below.

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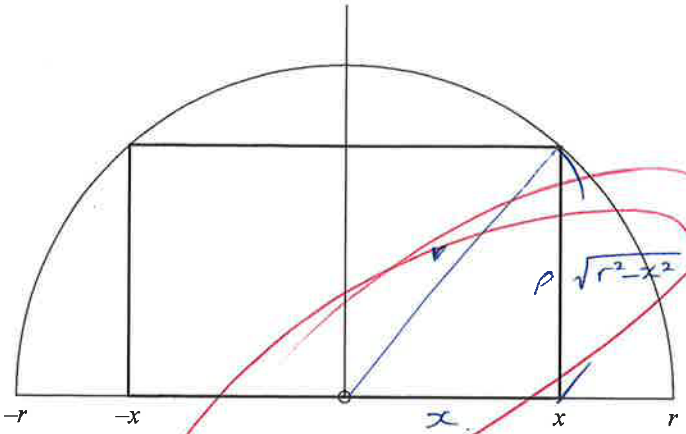

$r^2 = x^2 + $

$\frac{dA}{dx} = \frac{dP}{d}$

Show that the maximum possible area of such a rectangle occurs when $x = \frac{r}{\sqrt{2}}$. $\to r = x\sqrt{2}$.

You do not need to prove that your solution gives the maximum area.

You must use calculus and show any derivatives that you need to find when solving this problem.

$A = 2x \times \sqrt{r^2 - x^2}$

$\cancel{= 2x \times \sqrt{(x\sqrt{2})^2 - x^2}}$

$\cancel{= 2x\sqrt{2x^2 - x^2}}$

$\cancel{= 2x\sqrt{x^2}}$

$\cancel{= 2x^2}$

$\cancel{\frac{dA}{dx} = 4x}$

~~when maximum $\frac{dA}{dx} = 0$~~

~~$4x = 0$~~

~~$x = 0$~~

c/o

$\cancel{A = 2x \times \sqrt{r^2 - x^2}}$

$\cancel{= 2 \times \frac{r}{\sqrt{2}} \times \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2}}$

$\cancel{= \frac{2r}{\sqrt{2}} \times \sqrt{}}$

$A = 2x \times \sqrt{r^2 - x^2}$

$= 2x(r^2 - x^2)^{\frac{1}{2}}$

$\frac{dA}{dx} = 2(r^2 - x^2)^{\frac{1}{2}} + 2x\left(\frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}} \times (-2x)\right)$

$= 2(r^2 - x^2)^{\frac{1}{2}} + 2x(-x(r^2 - x^2)^{-\frac{1}{2}})$

$= 2(r^2 - x^2)^{\frac{1}{2}} - 2x^2(r^2 - x^2)^{-\frac{1}{2}}$

When maximum, $\frac{dA}{dx} = 0$.

$2(r^2 - x^2)^{\frac{1}{2}} - 2x^2(r^2 - x^2)^{-\frac{1}{2}} = 0$

$(r^2 - x^2)^{\frac{1}{2}} = x^2(r^2 - x^2)^{-\frac{1}{2}}$

$r^2 - x^2 = x^4(r^2 - x^2)^{-1}$

$r^2 - x^2 = \frac{x^4}{r^2 - x^2}$

nc

$(r^2 - x^2)^2 = x^4$

nc

$r^2 - 2rx + x^4 = x^4$

$r^2 - 2rx = 0 \to r(r - 2x) = 0$

$r - 2x = 0$

$-2x = -r$

$2x = r$

$x = \frac{r}{2}$

$(x-1)^2$

M6

QUESTION THREE

ASSESSOR'S USE ONLY

(a) Differentiate $y = x\ln(3x - 1)$.

You do not need to simplify your answer.

$\frac{dy}{dx} = \ln(3x-1) + x \times \frac{3}{3x-1}$

$= \ln(3x-1) + \frac{3x}{3x-1}$

(b) Find the gradient of the curve $y = \frac{1}{x} - \frac{1}{x^2}$ at the point $\left(2, \frac{1}{4}\right)$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = x^{-1} - x^{-2}$

$\frac{dy}{dx} = -x^{-2} + 2x^{-3}$

When $x=2$, $\frac{dy}{dx} = -(2)^{-2} + 2(2)^{-3}$

$= -0.25 + 0.25$

$= 0$

$\frac{dy}{dx} = 0$ at point ~~when~~ $\left(2, \frac{1}{4}\right)$

(c) The graph below shows the function $y = f(x)$.

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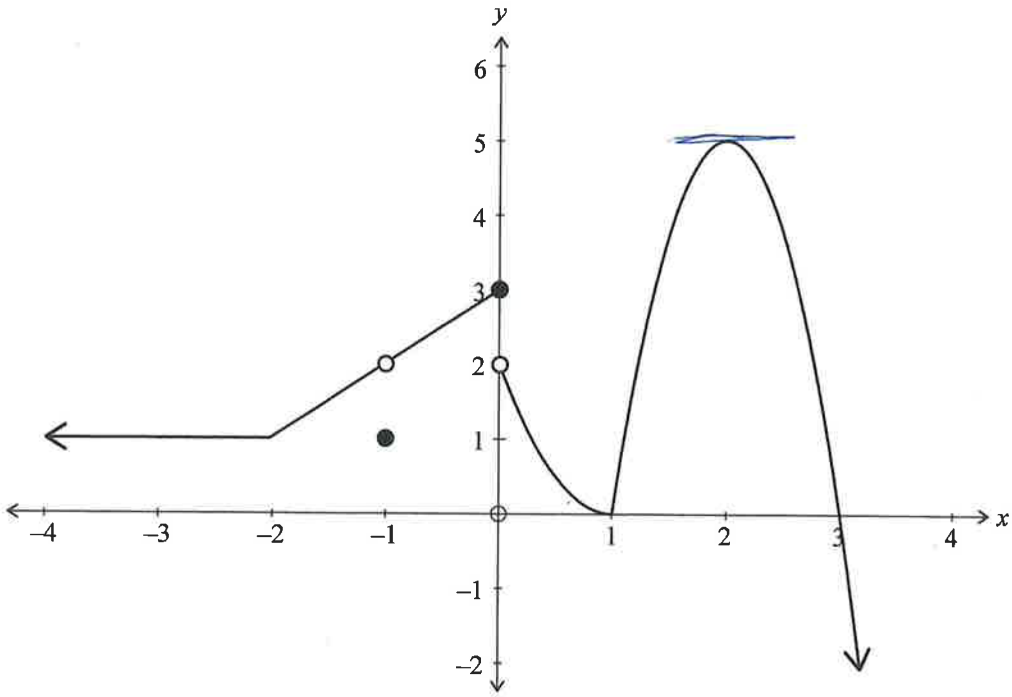


For the function above:

(i) Find the value(s) of x that meet the following conditions:

(1) $f'(x) = 0$: $x < -2$, $x = 2$

(2) $f(x)$ is continuous but not differentiable: $x = -2$, $x = 1$

(3) $f(x)$ is not continuous: $x = -1$, $x = 0$

(4) $f''(x) < 0$: $1 < x$

concave down.

(ii) What is the value of $\lim_{x \to -1} f(x)$? 2

State clearly if the value does not exist. does not exist.

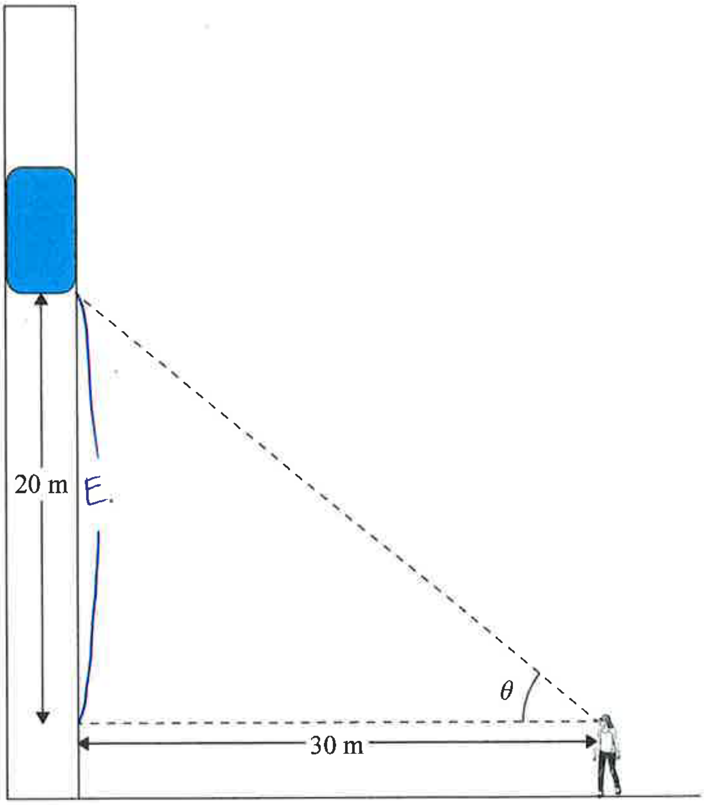
$\frac{dE}{dt} = 2$

(d) A building has an external elevator. The elevator is rising at a constant rate of 2 m s^{-1}.

Sarah is stationary, watching the elevator from a point 30 m away from the base of the elevator shaft.

Let the angle of elevation of the elevator floor from Sarah's eye level be θ.

www.alibaba.com/product-detail/Sicher-external-elevator_60136882005.html



$\frac{d\theta}{dt}$ $\quad$ $E=20$.

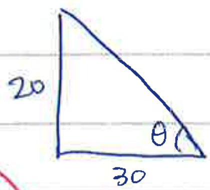
Find the rate at which the angle of elevation is increasing when the elevator floor is 20 m above Sarah's eye level.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{d\theta}{dt} = \frac{dE}{dt} \times \frac{d\theta}{dE}$ $\qquad$ $\tan\theta = \frac{E}{30}$

$= 2 \times \frac{\cos^2\theta}{30}$ $\qquad$ $E = 30\tan\theta$.

When $E=20$, $\qquad$ $\frac{dE}{d\theta} = 30\sec^2\theta$



$\tan\theta = \frac{20}{30}$ $\qquad$ $= 30(\sec\theta)^2$

$\theta = \tan^{-1}\frac{20}{30}$ $\qquad$ $= 30\left(\frac{1}{\cos\theta}\right)^2$

$= 33.69°$ $\qquad$ $= \frac{30}{\cos^2\theta}$

$\frac{d\theta}{dt} = 2 \times \frac{(\cos 33.69)^2}{30}$

$= 0.0462 \quad (4dp)$

$\therefore \frac{d\theta}{dt} = 0.0462°$ per second

ASSESSOR'S USE ONLY

(e) For the function $y = e^x \cos kx$:

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = e^x \cos kx + e^x \times (-\sin kx) \times k$$
$$= e^x \cos kx - ke^x \sin kx$$

$$\frac{d^2y}{dx^2} = (e^x \cos kx - ke^x \sin kx) - (ke^x \sin kx + ke^x \cos kx \times k)$$
$$= (e^x \cos kx - ke^x \sin kx) - (ke^x \sin kx + k^2 e^x \cos kx)$$
$$= e^x \cos kx - ke^x \sin kx - ke^x \sin kx - k^2 e^x \cos kx$$
$$= (1-k^2)(e^x \cos kx) - 2ke^x \sin kx$$

(ii) Find all the value(s) of k such that the function $y = e^x \cos kx$ satisfies the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \text{ for all values of } x.$$

$$e^x \cos kx - 2ke^x \sin kx - k^2 e^x \cos kx - 2(e^x \cos kx - ke^x \sin kx) + 2(e^x \cos kx) = 0$$

ME

$$e^x \cos kx \cancel{- 2ke^x \sin kx} + k^2 e^x \cos kx \cancel{- 2e^x \cos kx} \cancel{+ 2ke^x \sin kx} \cancel{+ 2e^x \cos kx} = 0$$
$$e^x \cos kx + k^2 e^x \cos kx = 0$$
$$(1+k^2)e^x \cos kx = 0$$
$$1 + k^2 = 0 \quad \text{CON}$$
$$k^2 = -1$$
$$k = \sqrt{-1}$$

M

M6

Annotated Exemplars for 91578 Differentiation 2017

Excellence exemplar

Subject:	Level 3 Calculus	Standard:	91578	Total score:	20

Q	Grade score	Annotation
1	E8	This response provides evidence for E8 because the candidate has provided a perfect response for part 1e, including a correct first derivative found by applying the quotient rule and then two correct relationships between a and b. To complete the solution to the problem, the candidate has correctly solved these two equations involving a and b simultaneously. This is a strong candidate because all of the five parts of the question have been completed accurately.
2	M6	The candidate achieved M6 for this question because part 2e was not correctly completed. The model for the rectangle inscribed in a semi-circle was found successfully and also differentiated correctly with the product rule. However more than one algebraic error has been made in the attempt to solve the first derivative equal to zero. Therefore the candidate scored r for the correct model and first derivative. The candidate successfully completed both of the merit questions, part 2c and 2d, so their question grade score was M6
3	M6	The candidate correctly identified all the values of x required for the features of the piecewise function in question part 3c(i) as well as the limit required in part 3c(ii). They gained M6 rather than M5, because they were also successful in finding the first and second derivatives of $y = e^x coskx$ that were required in part 3e(i). They were not awarded E7 for part 3e because they made two minor errors, a sign error in the second line and then in the final solution where they only gave one of the two possible solutions of the consistently formed equation. They were not awarded the r for part 3d because they failed to realise that the units for angles required in the question were radians not degrees.