3

SUPERVISOR'S USE ONLY

91578

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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2017

91578 Apply differentiation methods in solving problems

9.30 a.m. Thursday 23 November 2017
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Achievement

TOTAL 12

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QUESTION ONE

(a) Differentiate $y = \sqrt{x} + \tan(2x)$.

$y = x^{1/2} + \tan 2x$

$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + 2\sec^2 2x$

(b) Find the gradient of the tangent to the curve $y = \frac{e^{2x}}{x+2}$ at the point where $x = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

~~$y = e^{2x}(x+2)^{-1}$~~

$\frac{dy}{dx} = \frac{2e^{2x}(x+2) - 1 \times e^{2x}}{(x+2)^2}$

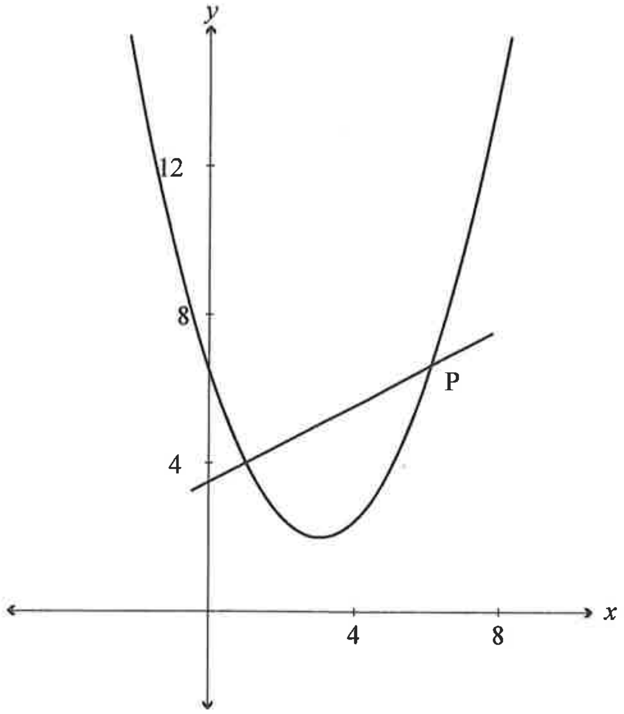
@ $x = 0$

$\frac{dy}{dx} = \frac{2e^{2(0)}(0+2) - e^{2(0)}}{(0+2)^2}$

$= \frac{4-1}{4}$

$= 3/4$ ← gradient at $x = 0$

(c) The normal to the parabola $y = 0.5(x - 3)^2 + 2$ at the point (1,4) intersects the parabola again at the point P.



Find the x-coordinate of point P.

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = 0.5(x-3)^2 + 2$

$\frac{dy}{dx} = 1(x-3)$

$= (x-3)$

@ $x = 1$

$\frac{dy}{dx} = 1-3$

$= -2$

Gradient of normal $= \frac{1}{2}$

Equation of normal:

$y - y_1 = m(x - x_1)$

$y - 4 = \frac{1}{2}(x-1)$

$y = \frac{1}{2}x - \frac{1}{2} + 4$

$y = \frac{1}{2}x + 3\frac{1}{2}$

$x = \frac{-\frac{7}{2} \pm \sqrt{(\frac{7}{2})^2 - (4 \times 0.5 \times 0)}}{2 \times 0.5}$

$= -\frac{7}{2} \pm \sqrt{12.25}$

$x = 0, -7$

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(d) A curve is defined parametrically by the equations $x = \sqrt{t+1}$ and $y = \sin 2t$.

Find the gradient of the tangent to the curve at the point when $t = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dx}{dt} = \frac{1}{2}(t+1)^{-\frac{1}{2}}$ $\quad \frac{dy}{dt} = 2\cos(2t)$

$= \frac{1}{2\sqrt{t+1}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 2\cos(2t) \times 2\sqrt{t+1}$

@ $t = 0$

$\frac{dy}{dx} = 2\cos(2\times0) \times 2\sqrt{0+1}$

$= 2 \times 2$

$= 4 \leftarrow$ gradient of tangent at $t = 0$

I1

(e) Find the values of $\underline{a}$ and $\underline{b}$ such that the curve $y = \dfrac{ax-b}{x^2-1}$ has a turning point at (3,1).

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You must use calculus and show any derivatives that you need to find when solving this problem.

$$y = \frac{ax-b}{x^2-1}$$

$$\frac{dy}{dx} = \frac{a(x^2-1) - 2x(ax-b)}{(x^2-1)^2}$$

$\cancel{ax^2 - a - 2ax^2 + 2bx}$

$$= \frac{ax^2 - a - 2ax^2 + 2bx}{(x^2-1)^2}$$

$$\frac{dy}{dx}$$

$$1 = \frac{3a-b}{3^2-1}$$

$8 \times 1 = 3a - b$

$3a - b = 8$

$3a - 8 = b$

M6

QUESTION TWO

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(a) Differentiate $y = 2(x^2 - 4x)^5$.

You do not need to simplify your answer.

$\frac{dy}{dx} = 10(x^2-4x)^4 \times (2x-4)$

(b) The percentage of seeds germinating depends on the amount of water applied to the seedbed that the seeds are sown in, and may be modelled by the function:

$$P(w) = 96\ln(w + 1.25) - 16w - 12$$

where P is the percentage of seeds that germinate and
w is the daily amount of water applied (litres per square metre of seedbed), with $0 \le w \le 15$.

Find the amount of water that should be applied daily to maximise the percentage of seeds germinating.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dP}{dw} = \frac{96}{(w+1.25)} - 16$

$\frac{96}{(w+1.25)} - 16 = 0$

$96 = 16(w+1.25)$

$16w = 96 - 20$

$16w = 76$

$w = 4.75$ litres daily to maximise percentage of seeds germinating.

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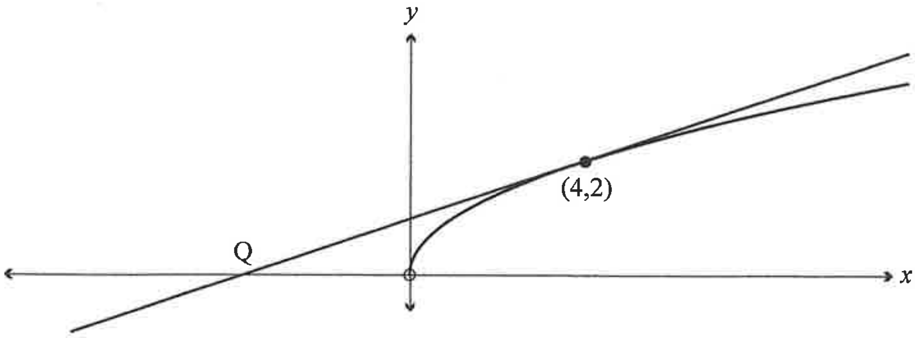
(c) The tangent to the curve $y = \sqrt{x}$ is drawn at the point (4,2).



Find the co-ordinates of the point Q where the tangent intersects the x-axis.

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

@ $x = 4$

$\frac{dy}{dx} = \frac{1}{2\sqrt{4}}$

$= \frac{1}{4}$

Equation of tangent:

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{4}x - 1$

$y = \frac{1}{4}x + 1$

@ $x = 0$

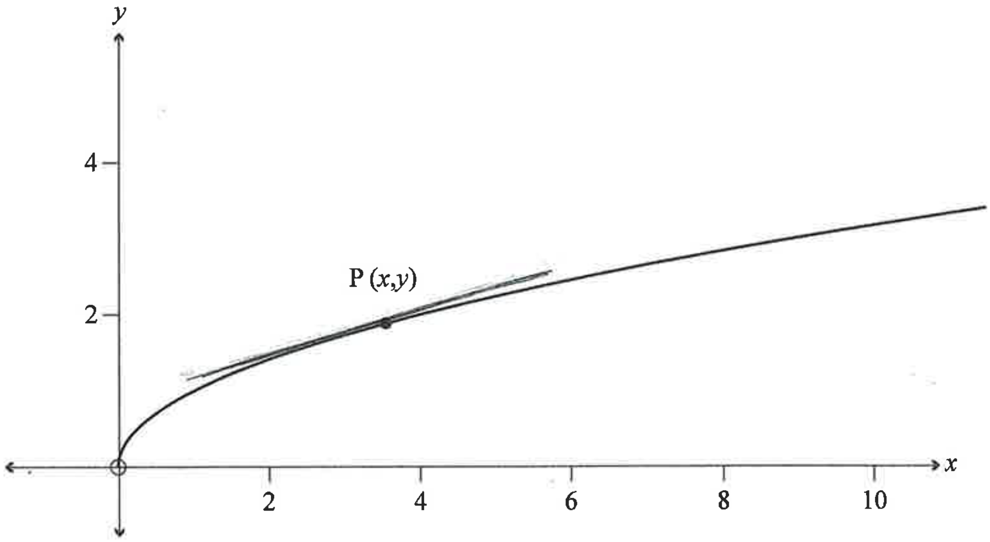
$y = \frac{1}{4}(0) + 1$

$y = 1$

Q = (0,1)

co-ordinates of Q

(d) Find the coordinates of the point P (x,y) on the curve $y = \sqrt{x}$ that is closest to the point (4,0).



You do not need to prove that your solution is the minimum value.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

@ $x = 4$

$\frac{dy}{dx} = \frac{1}{2\sqrt{4}}$

$= \frac{1}{4}$

$y - 0 = \frac{1}{4}(x-4)$

$y = \frac{1}{4}x - 1$

$x = \frac{1 \pm \sqrt{-1^2 - 4 \times \frac{1}{4} \times 0}}{2 \times \frac{1}{4}}$

$= \frac{1 \pm \sqrt{1}}{0.5}$

$x = 2$

@ $x = 2$ $\quad y = \sqrt{2}$

$= 1.414$

$P(2, 1.414)$

(e) A rectangle is inscribed in a semi-circle of radius r, as shown below.

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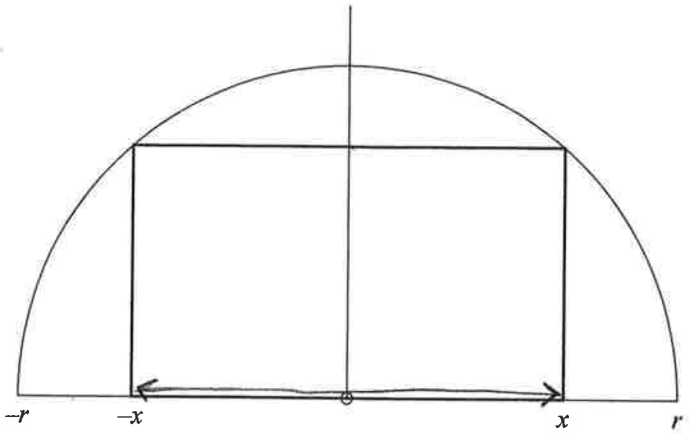

Show that the maximum possible area of such a rectangle occurs when $x = \frac{r}{\sqrt{2}}$.

You do not need to prove that your solution gives the maximum area.

You must use calculus and show any derivatives that you need to find when solving this problem.

Area of a circle $= \frac{1}{2}\pi r^2$

Area of rectangle $= b \times h$ ~~$b = 2x - 2r$~~ $b = 2r - 2x$

~~$= (2x - 2r)h$~~ $h = r - x$

$= (r - x)(r - x)$

$= r^2 - rx + xr + x^2$

$= r^2 + x^2 - 2rx$

$\frac{dA}{dr} = \pi r$

A4

ASSESSOR'S USE ONLY

QUESTION THREE

(a) Differentiate $y = x\ln(3x - 1)$.

You do not need to simplify your answer.

$\frac{dy}{dx} = \frac{x}{3x-1} \times 3$

$\frac{dy}{dx} = \frac{3x}{3x-1}$

(b) Find the gradient of the curve $y = \frac{1}{x} - \frac{1}{x^2}$ at the point $\left(2, \frac{1}{4}\right)$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = x^{-1} - x^{-2}$

$\frac{dy}{dx} = -x^{-2} + 2x^{-3}$

$= -\frac{1}{x^2} + \frac{2}{x^3}$

@ $x = 2$

$\frac{dy}{dx} = -2^{-2} + 2(2)^{-3}$

$= 0.25 + 0.25$

$= 0.5 \leftarrow$ gradient of tangent

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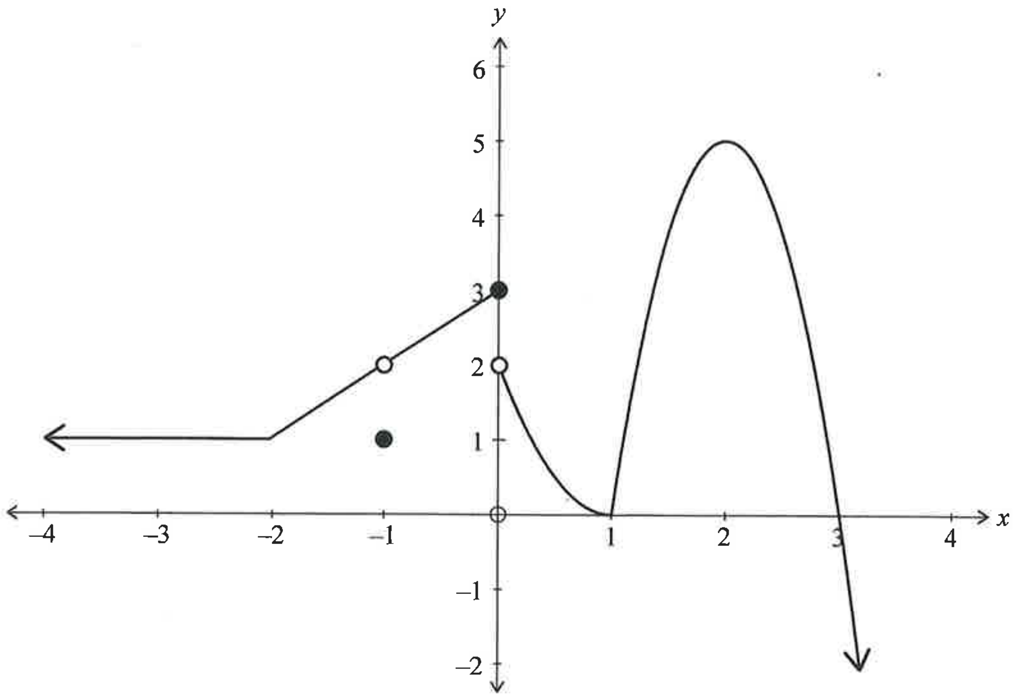
(c) The graph below shows the function $y = f(x)$.



For the function above:

(i) Find the value(s) of x that meet the following conditions:

(1) $f'(x) = 0$: $x = 1, 2$

(2) $f(x)$ is continuous but not differentiable:

(3) $f(x)$ is not continuous: $-2, 1$

(4) $f''(x) < 0$: $1 < x < 3$

(ii) What is the value of $\lim_{x \to -1} f(x)$? does not exist.

State clearly if the value does not exist.

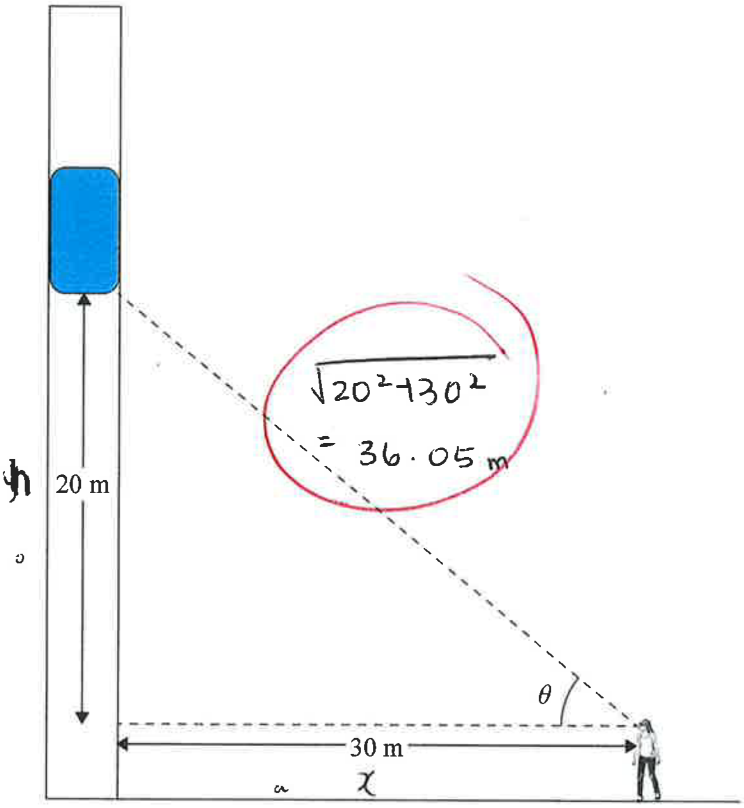
$\frac{dh}{dt} = 2$

(d) A building has an external elevator. The elevator is rising at a constant rate of 2 m s^{-1}.

Sarah is stationary, watching the elevator from a point 30 m away from the base of the elevator shaft.

Let the angle of elevation of the elevator floor from Sarah's eye level be θ.

www.alibaba.com/product-detail/Sicher-external-elevator_60136882005.html



Find the rate at which the angle of elevation is increasing when the elevator floor is 20 m above Sarah's eye level.

$\cancel{\frac{d\theta}{dt} = \frac{dh}{dt} \times \frac{d\theta}{dh}}$ $\frac{d\theta}{dh} = \frac{d\theta}{dt} \times \frac{dt}{dh}$

You must use calculus and show any derivatives that you need to find when solving this problem.

$\tan\theta = \frac{30}{20} \rightarrow \tan\theta = \frac{30}{h}$ $\tan\theta = \frac{30}{h}$

$\tan\theta = \frac{30}{h}$

$h = \frac{\tan\theta}{30}$

$\cancel{\frac{dh}{d\theta} = \frac{30\sec^2\theta}{900}}$

$\cancel{\frac{d\theta}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dh} \times \frac{dh}{dt}}$

$\cancel{= \frac{900}{30\cos^2\theta} \times 2}$

$\cancel{\frac{d\theta}{dt} = \frac{1800\cos^2\theta}{30}}$

(e) For the function $y = e^x \cos kx$:

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx} = e^x(-\sin kx) \times k + e^x \cos kx$

$= -ke^x \sin kx + e^x \cos kx \rightarrow e^x(-k\sin kx + \cos kx)$

$\frac{d^2y}{dx^2} = -ke^x \cos kx \times k + e^x(-\sin kx) \times k$

$= -k^2 e^x \cos kx - ke^x \sin kx$

$= ke^x(-k\cos kx - \sin kx)$

(ii) Find all the value(s) of k such that the function $y = e^x \cos kx$ satisfies the equation

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ for all values of x.

$(-k^2e^x\cos kx - ke^x \sin kx) - 2(-ke^x \sin kx + e^x \cos kx) + 2(e^x \cos kx) = 0$

$-k^2e^x \cos kx - ke^x \sin kx + 2ke^x \sin kx - 2e^x \cos kx + 2e^x \cos kx = 0$

$ke^x(-k\cos kx - \sin kx) - 2(e^x(-k\sin kx + \cos kx) + 2(e^x \cos kx) = 0$

~~$e^x - 2e^x + 2e^x = 0$~~

$(-k\cos kx - \sin kx) - (-k\sin kx + \cos kx) + (2e^x \cos kx) = 0$

N2

Achievement exemplar

Subject:	Level 3 Calculus	Standard:	91578	Total score:	12

Q	Grade score	Annotation
1	M6	The candidate achieved M6 for this question because they successfully found the parametric derivative and gradient required in part 1d as well as providing partial evidence towards the solution of the excellence question, part 1e. The r was awarded in part 1e because the candidate successfully differentiated the function provided and also substituted the given point into the original equation to find one relationship between the pronumerals, a and b. No further evidence was provided towards the solution of the problem in part 1e.
2	A4	This question provides evidence for A4 because they have achieved a total of three u grades. In part 2a, the candidate has correctly applied the chain rule to the polynomial function to find the first derivative. In part 2b, the candidate has successfully found the derivative of a function involving logarithms and then correctly solved the derivative equal to zero to find the amount of water required to maximise the percentage of seeds germinating. The candidate has achieved some partial evidence in part 2c by finding the correct derivative of the square root function and by then substituting into it to find the gradient of the required tangent. They have not achieved the r for this question because they substituted $x = 0$ and found the y-axis intercept rather than substituting $y = 0$ to find the x-axis intercept as required by the problem. This candidate was not able to find either of the models required by the problems in parts 2d and 2e.
3	N2	The candidate achieved N2 in question three due to the correct derivative they found in part 3e(i). They did not apply the product rule in part 3a so were not able to find the required derivative for the function provided. In part 3b, the candidate did find the required derivative but made a careless sign error when substituting $x = 2$. As a result they did not find the correct gradient.