91579

915790

3

SUPERVISOR'S USE ONLY

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Level 3 Calculus, 2017

91579 Apply integration methods in solving problems

9.30 a.m. Thursday 23 November 2017
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply integration methods in solving problems.	Apply integration methods, using relational thinking, in solving problems.	Apply integration methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Merit

TOTAL 16

ASSESSOR'S USE ONLY

ASSESSOR'S USE ONLY

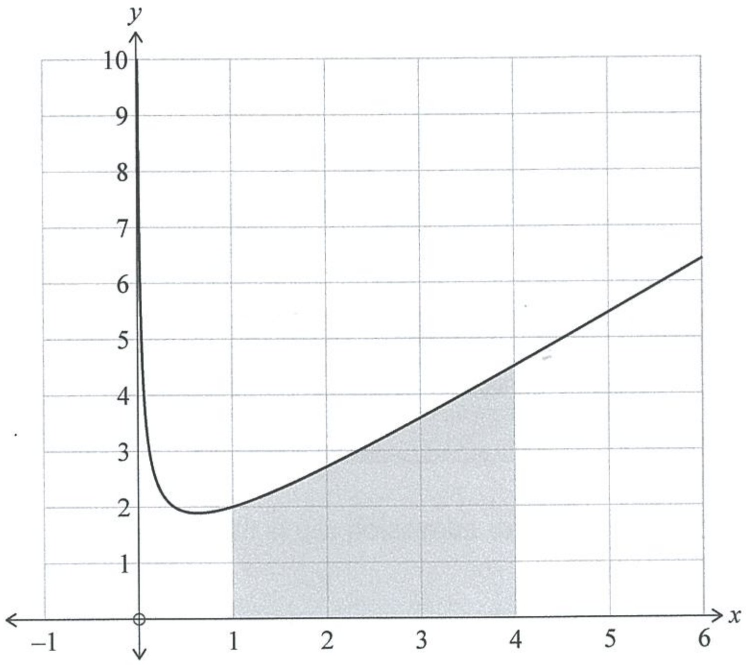
QUESTION ONE

(a) Find $\int 4\sec^2 2x\,dx$.

$= \frac{4}{2}\tan 2x + C$

$2\tan 2x + C$

(b) Use integration to find the area enclosed between the curve $y = \frac{x^2 + \sqrt{x}^{\frac{1}{2}}}{x}$ and the lines $y = 0$, $x = 1$, and $x = 4$ (the area shaded in the diagram below).



You must use calculus and show the results of any integration needed to solve the problem.

$y = x^1 + x^{-\frac{1}{2}}$

$\int_1^4 (\cancel{x^3+}\, x + x^{-\frac{1}{2}})\,dx$

$= \left[\frac{1}{2}x^2 + 2x^{\frac{1}{2}}\right]_1^4$

$= \left(\frac{1}{2}(4)^2 + 2(4)^{\frac{1}{2}}\right) - \left(\frac{1}{2}(1)^2 + 2(1)^{\frac{1}{2}}\right)$

$= (8 + 4) - \left(\frac{1}{2} + 2\right)$

$= (12) - (5/2)$

$\cancel{= \frac{21}{2}\text{ units}^2}$ $= 19/2$

$\cancel{= 10.5\text{ units}^2}$ $= 9.5\text{ units}^2$

(c) An object's acceleration is modelled by the function

$$\boxed{a(t)=1.2\sqrt{t}}$$

where a is the acceleration of the object, in m s^{-2}
and t is the time in seconds since the start of the object's motion.

If the object had a velocity of 7 m s^{-1} after 4 seconds, how far did it travel in the first 9 seconds of motion?

You must use calculus and show the results of any integration needed to solve the problem.

$a(t) = 1.2t^{\frac{1}{2}} \quad \therefore \quad v = \frac{1.2}{3/2} t^{3/2} + c$

$t = 4,\ v = 7 \Rightarrow 7 = 0.8(4)^{3/2} + c$

$7 = 6.4 + c$

$c = 0.6$

$v = 0.8t^{3/2} + 0.6$

~~t=9~~ $d = \frac{0.8}{5/2} t^{5/2} + 0.6t + c$

as $\cancel{d(t)}$ $d = 0,\ t = 0 \ \therefore c = 0$

$t=9$ $d = 0.32(9)^{5/2} + 0.6(9)$

$d = 77.76 + 5.4$

$= 83.16\text{ m}$ (4sf) $= 83.2\text{ m}$ (3sf)

(d) Find the value of k if $\int_0^k 3e^{2x}\,dx = 4$.

You must use calculus and show the results of any integration needed to solve the problem.

$\left[\frac{3}{2}e^{2x}\right]_0^k = 4$

$\left(\frac{3}{2}e^{2(k)}\right) - \left(\frac{3}{2}e^{2(0)}\right) = 4$

$\left(\frac{3}{2}e^{2k}\right) - \left(\frac{3}{2}\right) = 4$

$\frac{3}{2}e^{2k} = 4 + 3/2 = \frac{11}{2}$

$e^{2k} = \frac{11}{3}$

$2k = \ln\left|\frac{11}{3}\right|$

$k = 1.299/2$

$k = 0.6496$

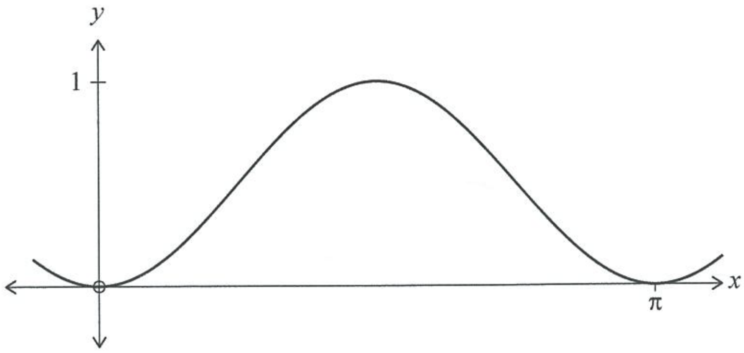
$\boxed{k = 0.650 \text{ (3sf)}}$

(e) The mean value of a function $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\text{Mean value} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Find the mean value of $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

Part of the graph of $y = \sin^2 x$ is shown below.



You must use calculus and show the results of any integration needed to solve the problem.

Mean value : $\dfrac{\int_0^\pi (\sin^2 x)\,dx}{\pi - 0}$ $\qquad \int \sin^2 x \, dx$

$= \frac{1}{2} \int -\cos 2x - 1$

$= \dfrac{\frac{1}{2}\int - \cos 2x - 1 \, dx}{\pi - 0}$

$= \dfrac{\frac{1}{2}\left[-\frac{1}{2}\sin 2x - x\right]_0^\pi}{\pi - 0}$

$= \dfrac{\frac{1}{2}\left[\left(-\frac{1}{2}\sin 2(\pi) - (\pi)\right) - \left(-\frac{1}{2}\sin 2(0) - (0)\right)\right]}{\pi - 0}$

$= \left\{\frac{1}{2}\left[(0 - \pi) - (0 - 0)\right]\right\} / (\pi - 0)$

$= \frac{-1}{2}\pi \;/ \pi$

$= -\frac{1}{2} = -0.5$ //

ASSESSOR'S USE ONLY

M6

QUESTION TWO

ASSESSOR'S USE ONLY

(a) Find $\int \frac{6}{2x-1} \mathrm{d}x$.

$\cancel{3}\int \frac{6}{2x-1} \, dx$

$= 3 \ln |2x-1| + c$

(b) Find $\int (2x-5)^4 \mathrm{d}x$.

$= \int (4x^2 - 20x + 25)(4x^2 - 20x + 25) \, dx$

$= \cancel{\int(16x^4 - 80} = (16x^4 - 80x^3 + 100x^2, - 80x^3 + 400x^2 - 500x,$

$+ 100x^2 - 500x + 625) \, dx$

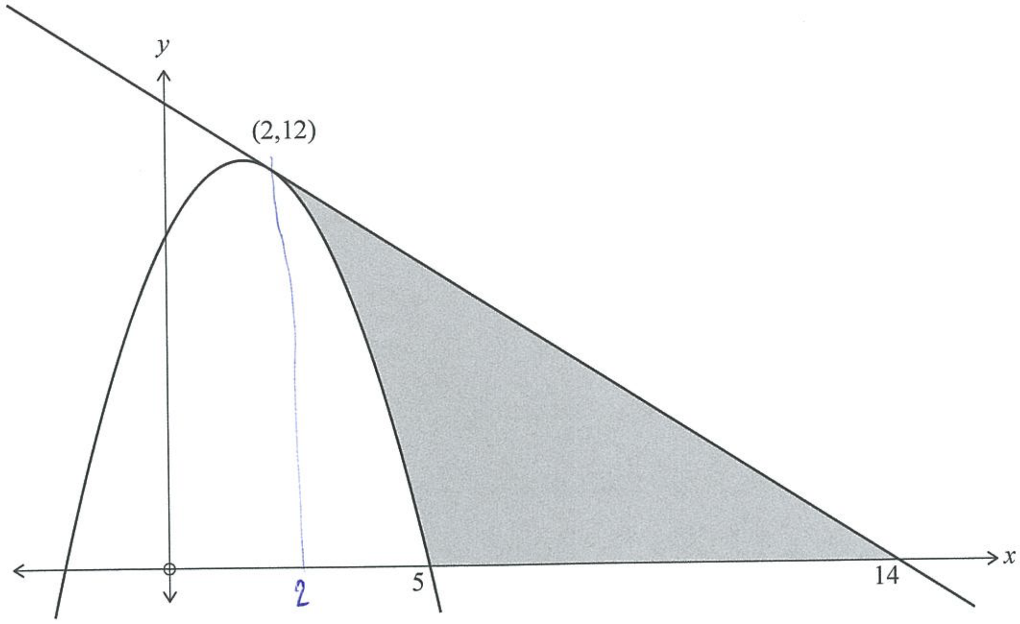
$= (16x^4 - 160x^3 + 600x^2 - 1000x + 625) \, dx$

$= \frac{16}{5}x^5 - 40x^4 + 200x^3 - 500x^2 + 625x + c$

$\frac{\frac{1}{5}(2x-5)^5}{x^2 - 5x} + c$

ASSESSOR'S USE ONLY

(c) The diagram below shows the curve $y = -x^2 + 3x + 10$, and the line $y = -x + 14$, which is the tangent to the curve at the point (2, 12).



Calculate the shaded area.

You must use calculus and show the results of any integration needed to solve the problem.

$A = \int_2^{14} [(-x+14)-(-x^2+3x+10)]\,dx$

$A = \int_2^{14} [x^2 - 4x + 4]\,dx$

$A = \left[\frac{1}{3}x^3 - \frac{4}{2}x^2 + 4x\right]_2^{14}$

$A = \left(\frac{1}{3}(14)^3 - 2(14)^2 + 4(14)\right) - \left(\frac{1}{3}(2)^3 - 2(2)^2 + 4(2)\right)$

$= \left(\frac{2744}{3} - 392 + 56\right) - \left(\frac{8}{3} - 8 + 8\right)$

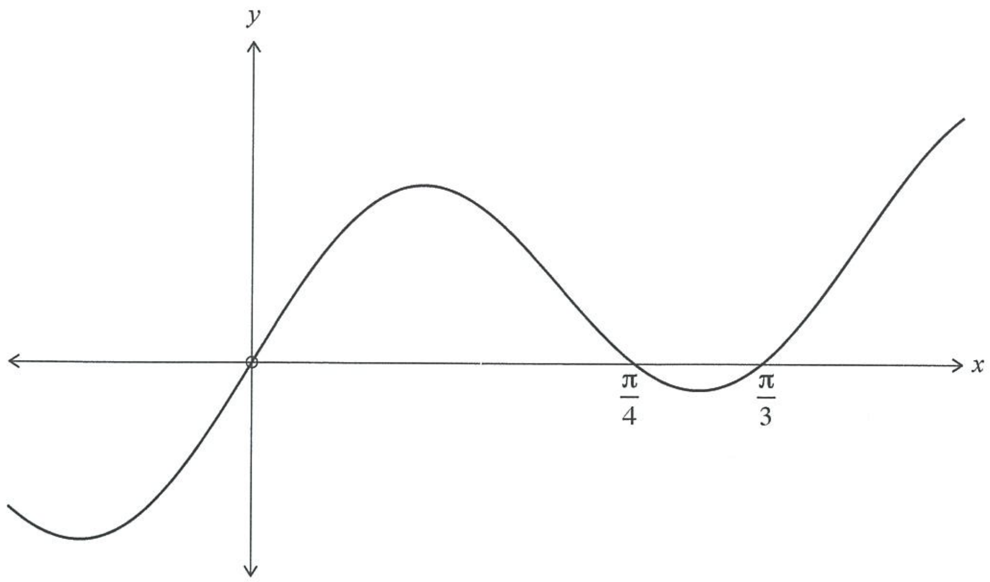
$= (1736/3) - (8/3)$

$= \cancel{577 \text{ units}^2}$

$A = 576 \text{ units}^2$ //

(d) Part of the graph of $\boxed{y = \sin 3x \cos 2x}$ is shown below.

ASSESSOR'S USE ONLY



Find the area enclosed between the curve $y = \sin 3x \cos 2x$ and the lines $y = 0$, $x = 0$, and $x = \frac{\pi}{4}$.

You must use calculus and show the results of any integration needed to solve the problem.

$A = \int_0^{\pi/3} (\sin 3x \cos 2x)\,dx$

$= \left(\int_0^{\pi/4} (\sin 3x \cos 2x)\,dx\right) - \left(\int_{\pi/4}^{\pi/3} (\sin 3x \cos 2x)\,dx\right)$

$= \left[-\frac{1}{3}\cos 3x \cdot \frac{1}{2}\sin 2x\right]_0^{\pi/4} - \left[-\frac{1}{3}\cos 3x \cdot \frac{1}{2}\sin 2x\right]_{\pi/4}^{\pi/3}$

$= \left\{\left(-\frac{1}{3}\cos 3\left(\frac{\pi}{4}\right) \cdot \frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)\right) - \left(-\frac{1}{3}\cos 3(0) \cdot \frac{1}{2}\sin 2(0)\right)\right\}$

$- \left\{\left(-\frac{1}{3}\cos 3\left(\frac{\pi}{3}\right) \cdot \frac{1}{2}\sin 2\left(\frac{\pi}{3}\right)\right) - \left(-\frac{1}{3}\cos 3\left(\frac{\pi}{4}\right) \cdot \frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)\right)\right\}$

$= \left[\left(\cancel{0.2357}\ 0.11785\right) - (0)\right]$

$- \left(\left(\cancel{\tfrac{1}{3}}\ 0.14434\right) - (0.11785)\right)$

$= \cancel{0.14434 \text{ units}^2}$

$= 0.09136 \text{ units}^2$

$A = 0.0914 \text{ units}^2$ (3sf)

(e) The acceleration of an object is modelled by the function $\boxed{a(t) = \frac{20 \ln t}{t}}$.

ASSESSOR'S USE ONLY

where a is the acceleration of the object in m s^{-2}
and t is the time in seconds since the start of the object's motion.

The object was moving with a velocity of 12 m s^{-1} when $t = 4$. $t=4,\ v=12$

Find the velocity of the object after 10 seconds. $t=10,\ v=?$

You must use calculus and show the results of any integration needed to solve the problem.

$v = \int \frac{20 \ln t}{t} = \int \frac{1}{t} \cdot 20 \ln t$

$= \ln t \times \frac{20}{t} + c$

$12 = \ln|4| \times \frac{20}{4} + c$

$c = 5.0685$

$v = \ln t \times \frac{20}{t} + 5.0685$

$v = \ln(10) \times \frac{20}{10} + 5.0685$

$v = 9.6737$

$= 9.67 ms^{-1}$ (3sf) 11

$\cancel{v = \int \frac{20 \ln t}{t} = \int \frac{1}{t} \cdot \frac{1}{20 \ln t}}$

$\cancel{= \ln t \times \frac{1}{20} \ln|\ln t| + c}$

$\cancel{12 = \ln|4| \times \frac{1}{20} \ln|\ln 4| + c}$

$\cancel{c = 11.977}$

$\cancel{v = \ln t \times \frac{1}{20} \ln|\ln t| + 11.977}$

$\cancel{v = \ln|10| \times \frac{1}{20} \ln|\ln|10|| + 11.977}$

A4

QUESTION THREE

ASSESSOR'S USE ONLY

(a) Find $\int\left(\frac{9}{x^4}+8e^{4x}\right)dx$.

$= \int(9x^{-4} + 8e^{4x})\,dx$

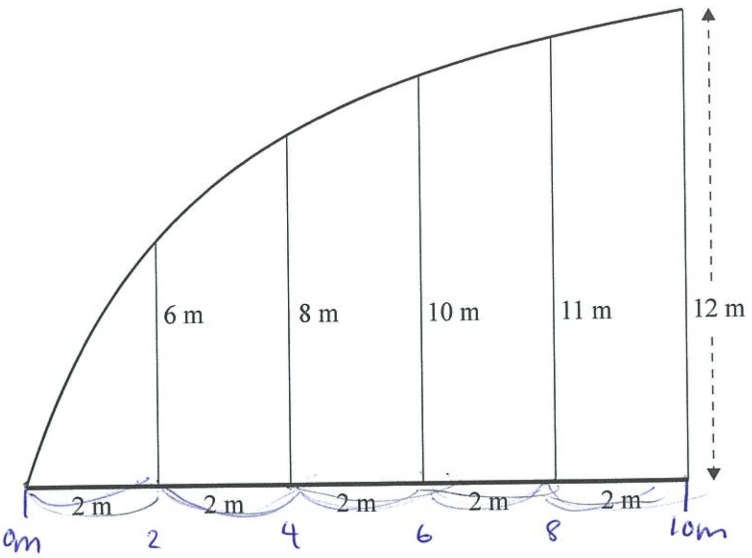
$= \frac{9}{-3}x^{-3} + \frac{8}{4}e^{4x} + C$

$= \frac{-3}{x^3} + 2e^{4x} + C$

Question Three continues on the following page.

ASSESSOR'S USE ONLY

(b) Julia wants to find an approximation of the area of a paved courtyard that she wishes to construct on her property. She takes some measurements and these are shown on the diagram below.



$n = 5$

Using these measurements, and the Trapezium rule, find an approximation of the area of paved courtyard.

x_n	0	2	4	6	8	10
y	0	6	8	10	11	12

$h = (10-0)/5 = 2$

$(\frac{1}{2}h)[y_0 + y_n + 2(y_1 + y_2 + y_3 + y_4)]$

$\Rightarrow (\frac{1}{2}(2))(0 + 12 + 2(6+8+10+11))$

$= (1)(12 + 2(35))$

$= 82\ m^2$

(c) Julia's friend Sarah believes that the equation of the curved border of the paved courtyard can be modelled by the function $y = \dfrac{15x - 15}{x + 2}$.

Use integration to find the area of the courtyard, shown in the diagram above.

You must use calculus and show the results of any integration needed to solve the problem.

$A = \int_1^{11} (15x - 15)/(x+2) = \int_1^{11} \dfrac{15(x+2) - 45}{x+2}$

$= \int_1^{11} 15 - \dfrac{45}{x+2}$

$= [15x - 45\ln|x+2|]_1^{11}$ –

$= (15(11) - 45\ln|(11)+2|) - (15(1) - 45\ln|1+2|)$

$= (165 - 115.42272) - (15 - 49.43755)$

$= (49.57728) - (-34.43755)$

$= 84.015$

$= 84.0$ m^2 ~~units^2~~ (3sf).

(d) Solve the differential equation $\frac{dy}{dx} = \frac{y}{\sqrt{x}}$, given that when $x = 4$, then $y = 1$.

ASSESSOR'S USE ONLY

You must use calculus and show the results of any integration needed to solve the problem.

$\Rightarrow \cancel{\frac{dy}{dx} = y \frac{1}{\sqrt{x}}}$ $\quad \frac{dy\sqrt{x}}{dx} = y$

$dy\sqrt{x} = y\,dx$

$\frac{dy}{y} = \frac{dx}{\sqrt{x}}$

$\frac{1}{y} \cdot dy = (x^{-\frac{1}{2}})\,dx$

$\ln|y| = 2x^{\frac{1}{2}} + C$ –

$\ln|1| = 2(4)^{\frac{1}{2}} + C$

$0 = 4 + C$

$C = -4$ –

$\ln|y| = 2\sqrt{x} - 4$

$y = e^{2\sqrt{x} - 4}$ # //

(e) y and t satisfy the differential equation $\frac{dy}{dt} = k\cos 0.5t \times e^{\sin 0.5t}$, $0 \le t \le 5$.

ASSESSOR'S USE ONLY

Given that when $t = 0$, $y = 8$, and that when $t = 2$, $y = 12$, find the value of y when $t = 5$.

You must use calculus and show the results of any integration needed to solve the problem.

$y = \frac{k}{0.5} \sin 0.5t \times \frac{1}{\frac{-1}{0.5}\cos 0.5t} e^{\sin 0.5t} + C$

$\sin 0.5t$

$\frac{-1}{0.5} \cos 0.5t$

$= 2k \sin 0.5t \times (\frac{-1}{2}\cos 0.5t) e^{\wedge \sin 0.5t} + C$

$8 = 2k \sin 0.5(0) \times (\frac{-1}{2}\cos 0.5(0)) e^{\wedge \sin 0.5(0)} + C$

$8 = 2k \cdot 0 \times (\frac{-1}{2}) + C$

$8 = C$

$\boxed{y = 2k \sin 0.5t \times (\frac{-1}{2}\cos 0.5t) e^{\wedge \sin 0.5t} + 8}$

$t = 2, \quad y = 12$

~~$12 = 2k \sin 0.5(2) \times (\frac{-1}{2}\cos 0.5(2)) e^{\wedge \sin 0.5(2)} + 8$~~

$12 = (2k \cdot 0.84147) \times (-0.62669) + 8$

$4 = 1.68294k \times (-0.62669)$

$\frac{4}{(-0.62669)} = 1.68294k$

$k = -3.7926 \qquad 2k = -7.5852$

$y = ?, \quad t = 5$

$y = \cancel{2k \sin 0.5} -7.5852 \sin 0.5(5) \times (\frac{-1}{2}\cos 0.5(5)) e^{\wedge \sin 0.5(5)} + 8$

$= -4.5395 \times (0.72878) + 8$

$y = 4.6917$

$= 4.69$ (3sf)

M6

<table>
<tr><th>Subject:</th><th colspan="2">Integration</th><th>Standard:</th><th>AS91579</th><th>Total score:</th><th>16</th></tr>
<tr><th>Q</th><th>Grade score</th><th colspan="5">Annotation</th></tr>
<tr><td>1</td><td>M6</td><td colspan="5">This question provides evidence for M6 because the candidate has gained 2r grades in part (c) and part (d) by:
In part (c)
- Correctly integrating the acceleration expression to find the velocity expression and correctly using the variables given to find the constant c of 0.6
- Correctly integrating the velocity expression to find the distance expression.
- Substituting in t = 9 to find d = 83.16m
In part (d)
- Correctly integrating the exponential expression
- Substituting in the limits k and 0 and equating to 4
- Using algebra to rearrange the expression and correctly calculate k = 0.6496</td></tr>
<tr><td>2</td><td>A4</td><td colspan="5">This question provides evidence for A4 because the candidate has gained 3 u grades in part (a), part(b) and part(c) by:
In part (a)
- Correctly integrating the expression
In part (b)
- Correctly integrating the expression
In part (c)
- They get a “u” grade because they set up a correct integral of the top line minus the bottom parabola and integrate this correctly. They do not get “r” because they use incorrect limits and do not calculate the correct required area.</td></tr>
<tr><td>3</td><td>M6</td><td colspan="5">This question provides evidence for M6 because the candidate has gained 2r grades in part (c) and part (d) by:
In part (c)
- Correctly splitting up the algebraic fraction into a form that can be integrated.
- Correctly integrating this expression
- Correctly substituting in the limits 1 and 14 to find the required area of 84.
In part (d)
- Correctly splitting the variables and integrating the differential equation
- Substituting in the variables given to find constant “c”</td></tr>
</table>

		- Correctly writing a solution to the differential equation.