91579

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SUPERVISOR'S USE ONLY

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Level 3 Calculus, 2017

91579 Apply integration methods in solving problems

9.30 a.m. Thursday 23 November 2017
Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply integration methods in solving problems.	Apply integration methods, using relational thinking, in solving problems.	Apply integration methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 24

ASSESSOR'S USE ONLY

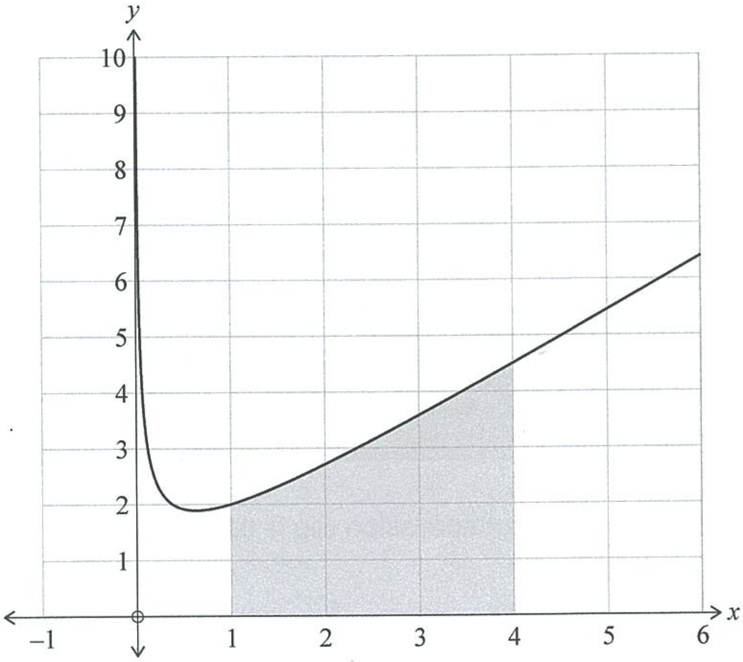
QUESTION ONE

ASSESSOR'S USE ONLY

(a) Find $\int 4\sec^2 2x\,dx$.

$= 2\tan 2x + C$

(b) Use integration to find the area enclosed between the curve $y = \dfrac{x^2 + \sqrt{x}}{x}$ and the lines $y = 0$, $x = 1$, and $x = 4$ (the area shaded in the diagram below).



You must use calculus and show the results of any integration needed to solve the problem.

$y = x + x^{-\frac{1}{2}}$

$A = \int_1^4 (x + x^{-\frac{1}{2}})\,dx$

$= \left[\frac{x^2}{2} + 2x^{\frac{1}{2}}\right]_1^4$

$= (8 + 4) - (\frac{1}{2} + 2)$

$= 9.5$ units^2

(c) An object's acceleration is modelled by the function

$$a(t) = 1.2\sqrt{t}$$

where a is the acceleration of the object, in m s^{-2}
and t is the time in seconds since the start of the object's motion.

If the object had a velocity of 7 m s^{-1} after 4 seconds, how far did it travel in the first 9 seconds of motion?

You must use calculus and show the results of any integration needed to solve the problem.

$v(t) = \int 1.2t^{\frac{1}{2}}\,dt \quad = \frac{2}{3}\cdot 1.2\cdot t^{\frac{3}{2}} + c \quad = \frac{4}{5}t^{\frac{3}{2}} + c$

when $t=4$, $V=7$

$7 = \frac{4}{5}(4)^{\frac{3}{2}} + c \qquad c = 0.6$

$v(t) = \frac{4}{5}t^{\frac{3}{2}} + \frac{3}{5}$

$s = \int_0^9 \left(\frac{4}{5}t^{\frac{3}{2}} + \frac{3}{5}\right) dt = \left[\frac{8}{25}t^{\frac{5}{2}} + \frac{3}{5}t\right]_0^9$

$= \frac{8}{25}(9)^{\frac{5}{2}} + 5.4 \quad = 83.16\,\text{m}$

A

(d) Find the value of k if $\int_0^k 3e^{2x}\,dx = 4$.

You must use calculus and show the results of any integration needed to solve the problem.

$\int_0^k 3e^{2x}\,dx = \left[\frac{3}{2}e^{2x}\right]_0^k = 4$

$\frac{3}{2}e^{2k} - \frac{3}{2} = 4$

$\frac{3}{2}e^{2k} = \frac{11}{2} \quad \Rightarrow e^{2k} = \frac{11}{3}$

$2k = \ln\left|\frac{11}{3}\right|$

$k = \frac{1}{2}\ln\left|\frac{11}{3}\right| = 0.6496$

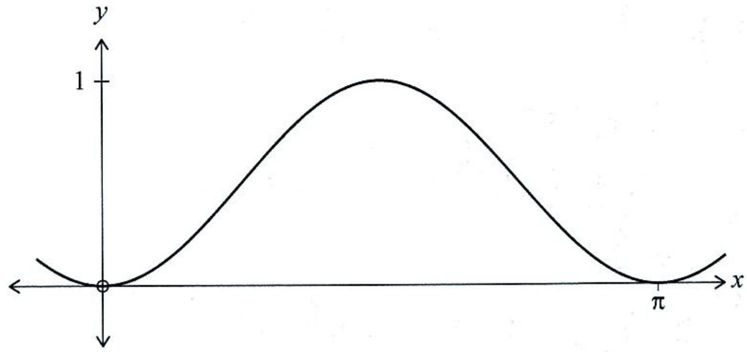
A

(e) The mean value of a function $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\text{Mean value} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Find the mean value of $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

Part of the graph of $y = \sin^2 x$ is shown below.



You must use calculus and show the results of any integration needed to solve the problem.

Mean value $= \dfrac{\int_0^\pi \sin^2 x\,dx}{\pi}$ $\quad \cancel{\int_0^\pi \cos 2x - 1}$ $\quad \cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$

$= \dfrac{\int_0^\pi \left(\frac{1}{2} - \frac{\cos 2x}{2}\right) dx}{\pi}$

$= \dfrac{\left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_0^\pi}{\pi} = \dfrac{\left(\frac{1}{2}\pi - 0\right) - (0 - 0)}{\pi}$

$= \dfrac{\frac{1}{2}\pi}{\pi} = \frac{1}{2}$

E8

QUESTION TWO

ASSESSOR'S USE ONLY

(a) Find $\int \frac{6}{2x-1}\,dx$.

$= 3\ln|2x-1| + C$

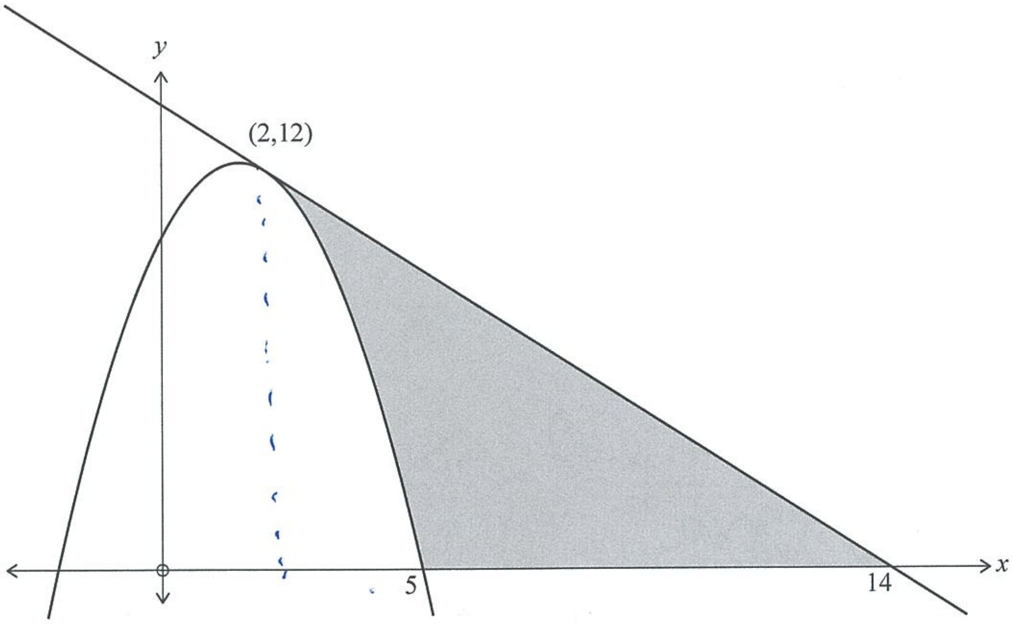
(b) Find $\int (2x-5)^4\,dx$.

$\frac{d\left((2x-5)^5\right)}{dx} = 5 \cdot 2(2x-5)^4$

$\therefore \int (2x-5)^4 dx = \frac{1}{10}(2x-5)^5 + C$

(c) The diagram below shows the curve $y = -x^2 + 3x + 10$, and the line $y = -x + 14$, which is the tangent to the curve at the point (2, 12).

ASSESSOR'S USE ONLY



Calculate the shaded area.

You must use calculus and show the results of any integration needed to solve the problem.

$A_s = A_t - A_p$ (A_s is shaded), (A_t is total), (A_p is area under part of parabola)

$A_t = \int_2^{14} (-x+14)\,dx = \left[-\frac{x^2}{2} + 14x\right]_2^{14} = (-98+196) - (-2+28)$

$= 72$ units^2

$A_p \quad \int_2^5 (-x^2+3x+10)\,dx = \left[-\frac{x^3}{3} + \frac{3}{2}x^2 + 10x\right]_2^5$

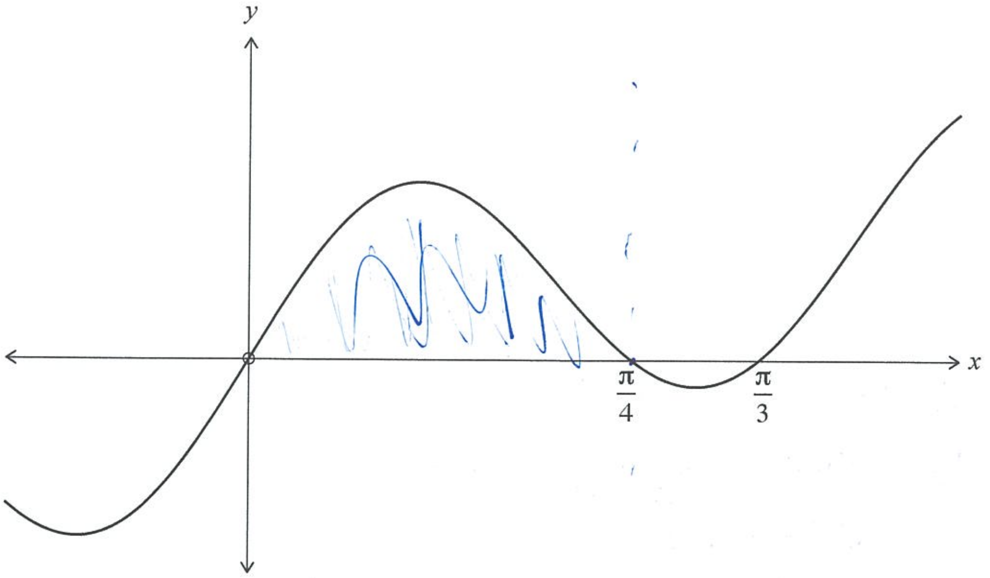
$= \left(-\frac{125}{3} + \frac{75}{2} + 50\right) - \left(-\frac{8}{3} + 6 + 20\right)$

$= 22.5$ units^2

$\therefore A_s = 72 - 22.5 = 49.5$ units^2

(d) Part of the graph of $y = \sin 3x \cos 2x$ is shown below.

ASSESSOR'S USE ONLY



Find the area enclosed between the curve $y = \sin 3x \cos 2x$ and the lines $y = 0$, $x = 0$, and $x = \frac{\pi}{4}$.

You must use calculus and show the results of any integration needed to solve the problem.

$\sin 3x \cos 2x = \frac{\sin 5x}{2} + \frac{\sin x}{2}$

$A = \int_0^{\frac{\pi}{4}} \left(\frac{\sin 5x}{2} + \frac{\sin x}{2}\right) dx = \left[\frac{-\cos 5x}{10} - \frac{\cos x}{2}\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\frac{\sqrt{2}}{2}}{10} - \frac{\frac{\sqrt{2}}{2}}{2}\right) - \left(-\frac{1}{10} - \frac{1}{2}\right)$

$= \frac{\sqrt{2}}{20} - \frac{5\sqrt{2}}{20} + \frac{1}{10} + \frac{1}{2}$

$= \frac{-4\sqrt{2}}{20} + \frac{12}{20} = \frac{12 - 4\sqrt{2}}{20} = \frac{3 - \sqrt{2}}{5}$

$= 0.317$ units^2

(e) The acceleration of an object is modelled by the function $a(t) = \dfrac{20 \ln t}{t}$.

ASSESSOR'S USE ONLY

where a is the acceleration of the object in m s^{-2}
and t is the time in seconds since the start of the object's motion.

The object was moving with a velocity of 12 m s^{-1} when $t = 4$.

Find the velocity of the object after 10 seconds.

You must use calculus and show the results of any integration needed to solve the problem.

$\int a(t)\,dt = \int \frac{20}{t} \ln(t)\,dt$ ~~$\ln|\ln(t)| + C$~~

~~Let $t = \ln|A|$ $\frac{dt}{dA} = \frac{1}{A}$ $dt = \frac{1}{A} dA$~~

~~$\ln(t) = \ln(\ln(A))$~~

~~$\Rightarrow \int \frac{20}{A\ln(A)} \cdot \ln(\ln(A))\,dA = 20 \ln|\ln(A)| + C$~~

Let $\ln|t| = A$ $t = e^A$ $\frac{dt}{dA} = e^A$ $dt = e^A\,dA$

$\int \frac{20}{e^A} \cdot A \cdot e^A\,dA = \int 20A\,dA$

$= 10A^2 + C$

$v(t) = 10(\ln t)^2 + C$

when $t = 4$, $v = 12$

$12 = 19.218 + C$

$C = -7.218$

$v(t) = 10(\ln t)^2 - 7.218$

when $t = 10$

$v(10) = 10(\ln 10)^2 - 7.218$

$= 45.8$ ms^{-1}

E8

QUESTION THREE

ASSESSOR'S USE ONLY

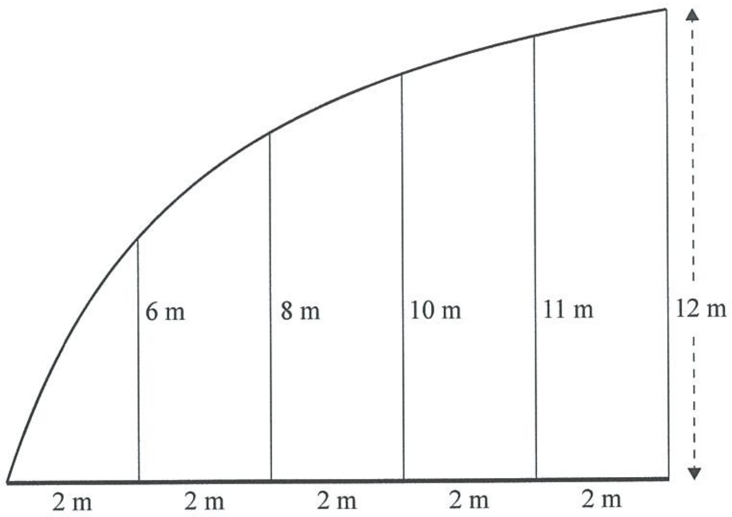
(a) Find $\int\left(\frac{9}{x^4}+8e^{4x}\right)dx$.

$= \int 9x^{-4} + 8e^{4x}\, dx$

$= -3e^{-3} + 2e^{4x} + C$

Question Three continues on the following page.

(b) Julia wants to find an approximation of the area of a paved courtyard that she wishes to construct on her property. She takes some measurements and these are shown on the diagram below.



Using these measurements, and the Trapezium rule, find an approximation of the area of paved courtyard.

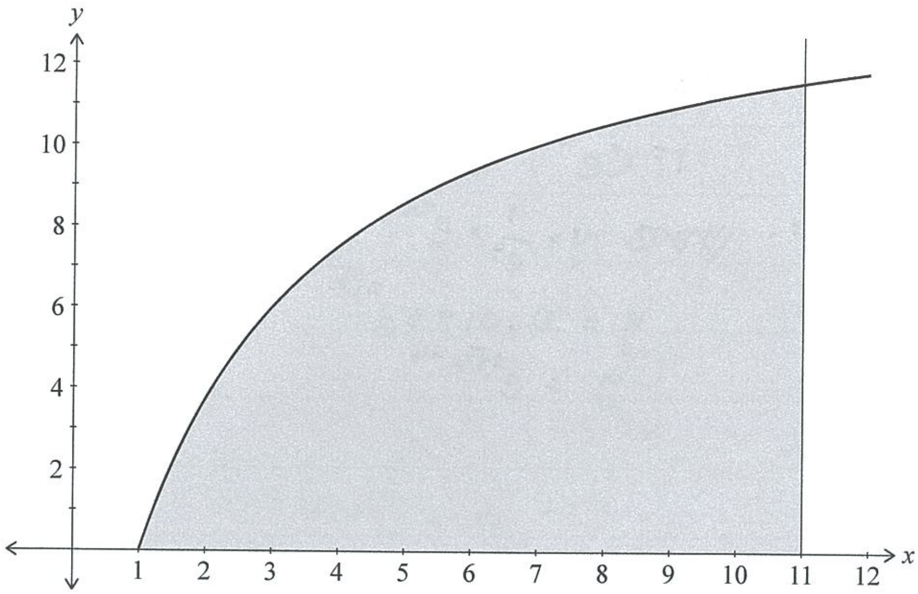
$h = \frac{10}{5} = 2$

$A = \int_0^{10} f(x)\,dx \approx \left[12 + 2(6+8+10+11)\right] = 82\,m^2$

ASSESSOR'S USE ONLY

(c) Julia's friend Sarah believes that the equation of the curved border of the paved courtyard can be modelled by the function $y = \dfrac{15x - 15}{x + 2}$.

ASSESSOR'S USE ONLY



Use integration to find the area of the courtyard, shown in the diagram above.

You must use calculus and show the results of any integration needed to solve the problem.

$$(x+2)\overline{)15x-15} \quad \text{quotient } 15; \quad 15x+30; \quad -45$$

$$\frac{15x-15}{x+2} = 15 - \frac{45}{x+2}$$

$$\int_1^{11} \left(15 - \frac{45}{x+2}\right) dx = \Big[15x - 45\ln|x+2|\Big]_1^{11}$$

$$= (165 - 115.42) - (15 - 49.437)$$

$$= 84.01 \text{ units}^2$$

(d) Solve the differential equation $\frac{dy}{dx} = \frac{y}{\sqrt{x}}$, given that when $x = 4$, then $y = 1$.

You must use calculus and show the results of any integration needed to solve the problem.

$\int \frac{1}{y}\,dy = \int x^{-\frac{1}{2}}\,dx$

$\ln\left|\frac{y}{c}\right| = 2x^{\frac{1}{2}}$ $\quad y = Ce^{2x^{\frac{1}{2}}}$

$1 = Ce^{4}$ $\quad C = 0.01831$

$\therefore$ ~~y=a~~ $y = \frac{1}{e^4} \cdot e^{2x^{\frac{1}{2}}}$

$y = 0.0183e^{2\sqrt{x}}$

$y = e^{2\sqrt{x} - 4}$

ASSESSOR'S USE ONLY

(e) y and t satisfy the differential equation $\frac{dy}{dt} = k\cos 0.5t \times e^{\sin 0.5t}$, $0 \leq t \leq 5$.

ASSESSOR'S USE ONLY

Given that when $t = 0$, $y = 8$, and that when $t = 2$, $y = 12$, find the value of y when $t = 5$.

You must use calculus and show the results of any integration needed to solve the problem.

$\int dy = k\int \cos(0.5t) \times e^{\sin(0.5t)}\, dt$

$y = k\left[2e^{\sin(0.5t)}\right] + C$

→ when $y = 8$, $t = 0$

$8 = k[2] + C$

$8 = 2k + C$ (1)

$C = 8 - 2k$

→ when $y = 12$, $t = 2$

$12 = k\left[2e^{\sin(1)}\right] + C$

$12 = k(4.63955) + C$

$12 = 4.63955k + 8 - 2k$

$k = 1.515$

$\therefore C = 4.969$

$\therefore y = 1.515\left(2e^{\sin(0.5t)}\right) + 4.969$

when $t = 5$

$y = 10.481$

E8

Subject:	Integration	Standard:	AS91579	Total score:	24

Q	Grade score	Annotation
1	E8	This question provides evidence for E8 because the candidate has gained 1**t** in part (e) by: - Correctly using a trigonometric identity to change $\sin^2 x$ into a form that can be integrated - Correctly integrating this expression - Substituting in the limits π and 0 into the expression - Simplifying the Mean Value expression to get ½.
2	E8	This question provides evidence for E8 because the candidate has gained 1**t** in part (e) by: - Correctly integrating the acceleration expression to get a velocity expression - Using the variables given to correctly find constant "c" - Substituting $t = 10$ to correctly find the velocity of 45.8 m/s
3	E8	This question provides evidence for E8 because the candidate has gained 1**t** in part (e) by: - Correctly integrating the differential equation - Using the initial variables given to find an expression for c in terms of k. - Using the other variables provided and this expression to correctly calculate k. - Uses the expression to correctly calculate c - Substituting $t = 5$ to correctly find $y = 10.48$.