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Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Level 3 Calculus 2024

91577 Apply the algebra of complex numbers in solving problems

Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

Show ALL working.

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Achievement

TOTAL 09

QUESTION ONE

(a) Given that $x + 3$ is a factor of $x^3 + px^2 + 5x - 12$, find the value of the constant p.

$x+3=0 \quad x=-3$

$(-3)^3 + p(-3)^2 + 5x-3 - 12 = 0$

$-27 + 9p - 27 = 0$

$p = 6$

(b) If $z = m\text{cis}\left(\frac{n\pi}{5}\right)$, where m and n are positive real constants, then find z^{15}, giving your answer in polar form, in terms of m and n.

$m = r$

$n = 15$

(c) Solve the following equation for x, in terms of k, where k is a positive real constant.

$4 - \sqrt{kx} = \sqrt{(kx+4)}$

~~$16 - kx = kx + 4$~~

$\sqrt{4^2 - (\sqrt{kx})^2} = \sqrt{(\sqrt{(kx+4)})^2}$

$16 - kx = kx + 4$

$2kx = 12$

$kx = 6$

$x = \frac{6}{k}$

(d) The locus described by $|z - \mathrm{i}| = |z + 1|$ is a straight line.

Find the gradient of that line.

Justify your answer.

let $z = x + iy$

$|x + (y-1)i| = (x+1) + iy$

$\sqrt{x^2 + (y-1)^2} = \sqrt{(x+1)^2 + y^2}$

$x^2 + y^2 - 2y + 1 = x^2 + 2x + 1 + y^2$

$-2y = 2x$

$y = -x$

(e) Consider the complex numbers $u = 2 + 3k\mathrm{i}$ and $v = 4 + 5k\mathrm{i}$, where k is a real constant.

Show that the complex number $w = \frac{u}{v}$ will **not** lie on the line $y = x$ in the Argand diagram, for any value of k.

$w = \frac{2+3ki}{4+5ki}$

QUESTION TWO

(a) Write the complex number $\frac{i}{2k+i}$ in the form $a + bi$, where a, b, and k are real numbers, giving your answer in terms of k.

$\frac{(i)(2k-i)}{(2k+i)(2k-i)} = \frac{2ki+1}{4k^2+1}$ $\quad \frac{2ki}{4k^2+1} + \frac{1}{4k^2+1} = \frac{1}{4k^2+1} + \frac{2k}{4k^2+1}i$

$\therefore a = \frac{1}{4k^2+1}$

$b = \frac{2k}{4k^2+1}$

(b) Find the value(s) of r so that the quadratic equation $2x^2 + (3 + 2r)x + 3 - 2r = 0$ has equal roots.

$2x^2 + (3+2r)x + 3 - 2r = 0$

$2x^2 + 3x + 2rx + 3 - 2r = 0$

$x^2 + \frac{3}{2}x + rx + \frac{3}{2} - r = 0$

$x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} + rx - r + \frac{24}{16} = 0$

$(x + \frac{3}{4})^2 + rx - r + \frac{15}{16} = 0$

$(x - \frac{3}{4})^2 + r(x-1) = \frac{15}{16}$

(c) Given that $\frac{w}{w+i} = 2 - i$, find $|w|$.

$\frac{w}{w+1} = 2-i$

$w = 2w+1 + (2-w)i$

$|w| = \sqrt{(2w+1)^2 + (2-w)^2}$

$|w| = \sqrt{5w^2+5}$

$|w| = w\sqrt{5} + \sqrt{5}$

(d) One solution of the equation $2z^3 + dz^2 + 140z - 200 = 0$ is $z = 6 - 2i$.

If d is real, find the value of d and the other two solutions of the equation.

$\because z = 6 - 2i$

$2(6-2i)^3 + d(6-2i)^2 + 140(6-2i) - 200 = 0$

$288 - 416i + d36 - d24i - 4d + 840 - 280i - 200 = 0$

$288 + 840 - 200 + 32d - 416i - d24i - 280i = 0$

$928 + 32d - 696i - d24i = 0$

$928 + 32d - 696 - 24d = 0$

$232 + 8d = 0$

$8d = -232$

$d = -29$

(e) The locus of a complex number z is described by

$$|z - 1 - 7i| = 2|z - 4 - 4i|$$

The complex number $u = 3 + di$ lies on this locus.

Find the Cartesian equation of the locus of z, giving your answer in the form $(x - a)^2 + (y - b)^2 = k$ and **also find** the complex number(s) u.

let $z = x + iy$

$|(x-1)+(y-7)i| = 2|(x-4)+(y-4)i|$

$\sqrt{(x-1)^2+(y-7)^2} = 4\sqrt{(x-4)^2+(y-4)^2}$

$(x-1)^2 + (y-7)^2 = (x-4)^2 + (y-4)^2$

$x^2-2x+1+y^2-14y+49 = x^2-8x+16+y^2-8y+16$

$-2x+8x+1-14y+8y+49 = 32$

$6x-6y = -18$

$x - y = -3$

$-y = -3 - x$

$y = 3 + x$

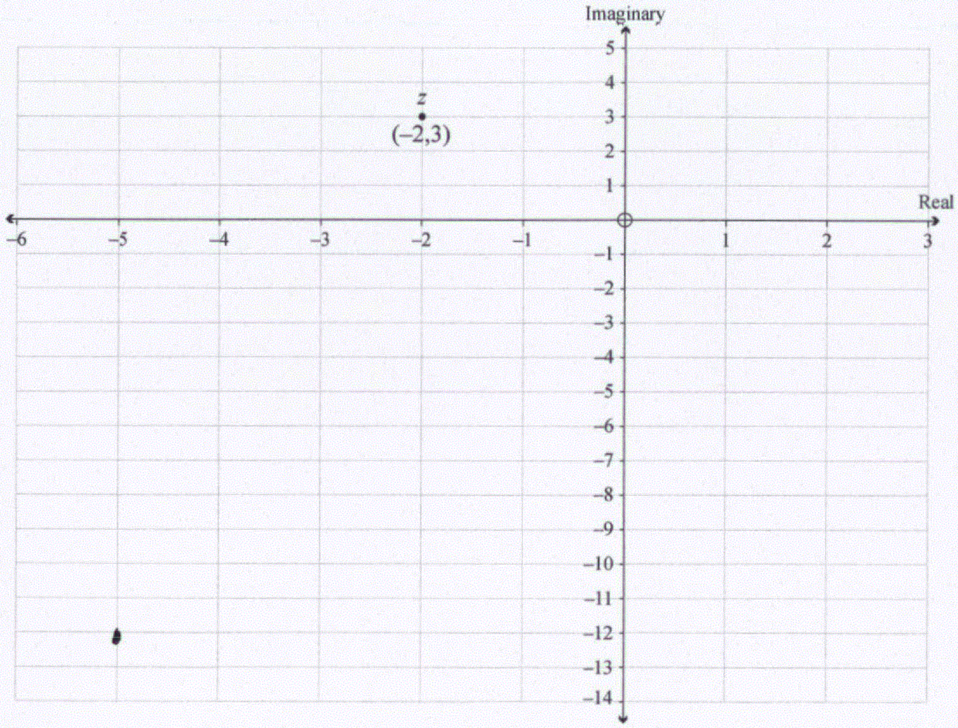
QUESTION THREE

(a) Write $\dfrac{\sqrt{2p}}{\sqrt{2p}-\sqrt{p}}$ in the form $a+\sqrt{a}$, where p and a are real constants.

$\dfrac{(\sqrt{2p})(\sqrt{2p}+\sqrt{p})}{2p-p} = \dfrac{2p+p\sqrt{2}}{p} = 2+\sqrt{2} \quad \therefore a = 2$

(b) In the Argand diagram below, the point (–2,3) represents the complex number z.

Show clearly, in the diagram below, the point representing $w = z^2$



Question Three continues on the next page.

(c) Find the value(s) of the real constant d, given that $z = 3 + d\mathrm{i}$ and $\overline{z} = 10dz^{-1}$

$z = 3 + d_i$ $\quad \bar{z} = 10dz^{-1} = \frac{10d}{z} \because z = 3 + d_i \quad \frac{10d}{3+d_i}$

$\frac{(10d)(3-d_i)}{(3+d_i)(3-d_i)} = \frac{30d - 10d^2 i}{9+d}$

(d) Solve the equation $z^4 + 81k^8 = 0$, where k is a real constant.

Give your solution(s) in polar form in terms of k.

(e) Given that $x + \frac{1}{x} = p$, where p is a real constant, then find the value of $x^3 + \frac{1}{x^3}$, giving your answers in terms of p.

Extra space if required.
Write the question number(s) if applicable.

QUESTION NUMBER

8.8

let $u = 2x$ $\frac{du}{dx} = 2$

$\frac{du}{2} = dx$

3 sec u

$5x^2 - 3 = u$

$10x = \frac{du}{dx}$ $\frac{du}{10x} = du$

81 23.4256

$(3x^4+4)^2$

$= 9x^8 + 24x^4 + 16$

u

3 sin(3x)

$\frac{du}{3}$

$\frac{4}{3}$ $\frac{2(-4x)}{4}$

8 8 sin(3x)+(cos(x))

91577

Achievement

Subject: Calculus

Standard: 91577

Total score: 9

Q	Grade score	Marker commentary
One	A3	1a correct substitution using factor theorem and correct response, u grade awarded. 1d correct algebraic manipulation to find the locus, but did not go on to answer the question posed, u grade awarded.
Two	A3	2a correct rearrangement using the conjugate, and identifies *a* and *b*, u grade awarded. 2d found *d* using the factor theorem but unable to find the third solution to the cubic equation, u grade awarded.
Three	A3	3a denominator rationalised and the expression simplified correctly and *a* identified, u grade awarded. 3b *w* calculated and clearly identified on the Argand diagram, u grade awarded.