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Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Level 3 Calculus 2024

91577 Apply the algebra of complex numbers in solving problems

Credits: Five

Achievement	Achievement with Merit	Achievement with Excellence
Apply the algebra of complex numbers in solving problems.	Apply the algebra of complex numbers, using relational thinking, in solving problems.	Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

Show ALL working.

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 21

QUESTION ONE

(a) Given that $x + 3$ is a factor of $x^3 + px^2 + 5x - 12$, find the value of the constant p.

(b) If $z = m\text{cis}\left(\frac{n\pi}{5}\right)$, where m and n are positive real constants, then find z^{15}, giving your answer in polar form, in terms of m and n.

(c) Solve the following equation for x, in terms of k, where k is a positive real constant.

$$4 - \sqrt{kx} = \sqrt{(kx + 4)}$$

(d) The locus described by $|z - \mathrm{i}| = |z + 1|$ is a straight line.

Find the gradient of that line.

Justify your answer.

$|z-i| = |z+i|$, $z = x+iy$

$|x+iy-i| = |x+iy+i|$

$|x+i(y-1)| = |x+i(y+1)|$

$\sqrt{x^2+(y-1)^2} = \sqrt{x^2+(y+1)^2}$

$x^2+(y-1)^2 = x^2+(y+1)^2$

$x^2+y^2-2y+1 = x^2+y^2+2y+1$

$-2y+1 = 2y+1$

$-4y+1 = 1$

$-4y = 0$

$y = 0$, purely real so gradient is ~~undefined~~ 0

$\cancel{x^2+(-1)^2 = x^2+1^2}$

$\cancel{x^2+1 = x^2+1}$

(e) Consider the complex numbers $u = 2 + 3k\mathrm{i}$ and $v = 4 + 5k\mathrm{i}$, where k is a real constant.

Show that the complex number $w = \frac{u}{v}$ will **not** lie on the line $y = x$ in the Argand diagram, for any value of k.

$u = 2+3ki \quad v = 4+5ki$

$W = \frac{u}{v}$, $= \frac{2+3ki}{4+5ki}$, $= \frac{(2+3ki)(4-5ki)}{(4+5ki)(4-5ki)}$

$= \frac{8+12ki-10ki-15k^2i^2}{16-25k^2i^2}$

$= \frac{8+2ki+15k^2}{16+25k^2}$

$= \frac{8+15k^2}{16+25k^2} + \frac{2ki}{16+25k^2}$

Real has to = imaginary to lie on $y=x$

as $\arg w = \left(\frac{\pi}{4}\right)$

$8+15k^2 = 2k$

$8-2k+15k^2=0$

$a=15, b=-2, c=8 \rightarrow b^2-4ac$

$(-2)^2 - 4\times15\times8$

$= 4-480$, $=-476$

$b^2-4ac<0$ so the equation has no roots so no value of k will lie on $y=x$ as real $\neq$ imaginary

QUESTION TWO

(a) Write the complex number $\frac{\mathrm{i}}{2k+\mathrm{i}}$ in the form $a + b\mathrm{i}$, where a, b, and k are real numbers, giving your answer in terms of k.

(b) Find the value(s) of r so that the quadratic equation $2x^2 + (3 + 2r)x + 3 - 2r = 0$ has equal roots.

(c) Given that $\dfrac{w}{w+\mathrm{i}} = 2 - \mathrm{i}$, find $|w|$.

(d) One solution of the equation $2z^3 + dz^2 + 140z - 200 = 0$ is $z = 6 - 2i$.

If d is real, find the value of d and the other two solutions of the equation.

$2z^3 + dz^2 + 140z - 200 = 0$

$z = 6-2i$

$2(6-2i)^3 + d(6-2i)^2 + 140(6-2i) - 200 = 0$

~~8250~~

$288 - 416i + d(32-24i) + 840 - 280i - 200 = 0$

$d(32-24i) = 696i$

$d = -29$

~~$2z^3 + dz^2 + 1$~~

$= 2z^3 - 29z^2 + 140z - 200$

other solution is $(6+2i)$

~~$(6-2i)(6+2i)$~~ $(z-(6+2i))(z-(6-2i))(2z+c)$

when $z=0$

$-(6+2i) \times -(6-2i) \times c = -200$

$40c = -200$

$c = -5$

$(2z-5)$, last solution , $2z - 5 = 0$

$2z = 5$

$z = \frac{5}{2}$

(e) The locus of a complex number z is described by

$$|z - 1 - 7i| = 2|z - 4 - 4i|$$

The complex number $u = 3 + di$ lies on this locus.

Find the Cartesian equation of the locus of z, giving your answer in the form $(x - a)^2 + (y - b)^2 = k$ and **also find** the complex number(s) u.

$|z-1-7i| = 2|z-4-4i|$

$z = x+iy$

$|x+iy-1-7i| = 2|x+iy-4-4i|$

$|x-1+i(y-7)| = 2|x-4+i(y-4)|$

$\sqrt{(x-1)^2+(y-7)^2} = 2\sqrt{(x-4)^2+(y-4)^2}$

$(x-1)^2+(y-7)^2 = 4(x-4)^2+(y-4)^2$

$(x^2-2x+1)+(y^2-14y+49) = 4[(x^2-8x+16)+(y^2-8y+16)]$

$x^2-2x+y^2-14y+50 = 4(x^2+y^2-8x-8y+32)$

$x^2-2x+y^2-14y+50 = 4x^2+4y^2-32x-32y+128$

$0 = 3x^2+3y^2-30x-18y+78$

$= x^2+y^2-10x-6y+26$

$= \cancel{(x+5)^2+(y+3)^2-8}$

$(x-5)^2+(y-3)^2-8$

$\cancel{(x+5)^2+(y+3)^2} = 8$

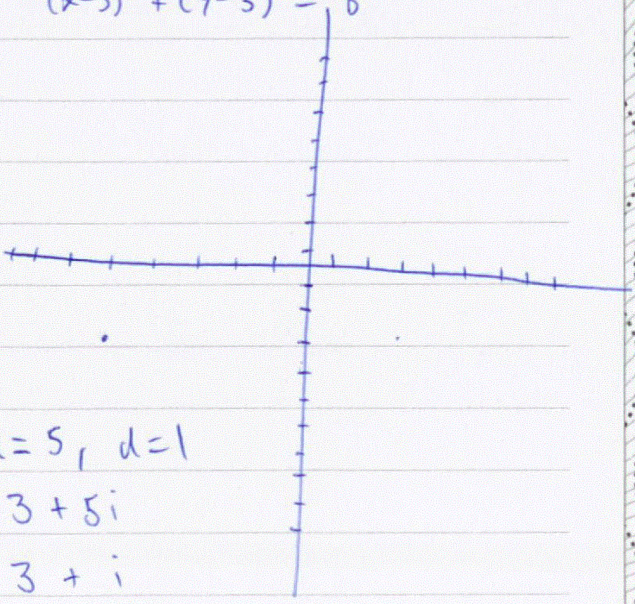
~~$u=3+di$~~

$\cancel{(3+5)^2+(d+3)^2=8}$

$\cancel{8^2+d^2+6d+9=8}$

$\cancel{56+d^2+6d+9=0}$

$\cancel{d^2+6d+65=0}$



$(x-5)^2+(y-3)^2-8$

$u=3+di$

$(3-5)^2+(d-3)^2=8$

$4+d^2-6d+9=8$

$d^2-6d+5=0$

$d=5,\ d=1$

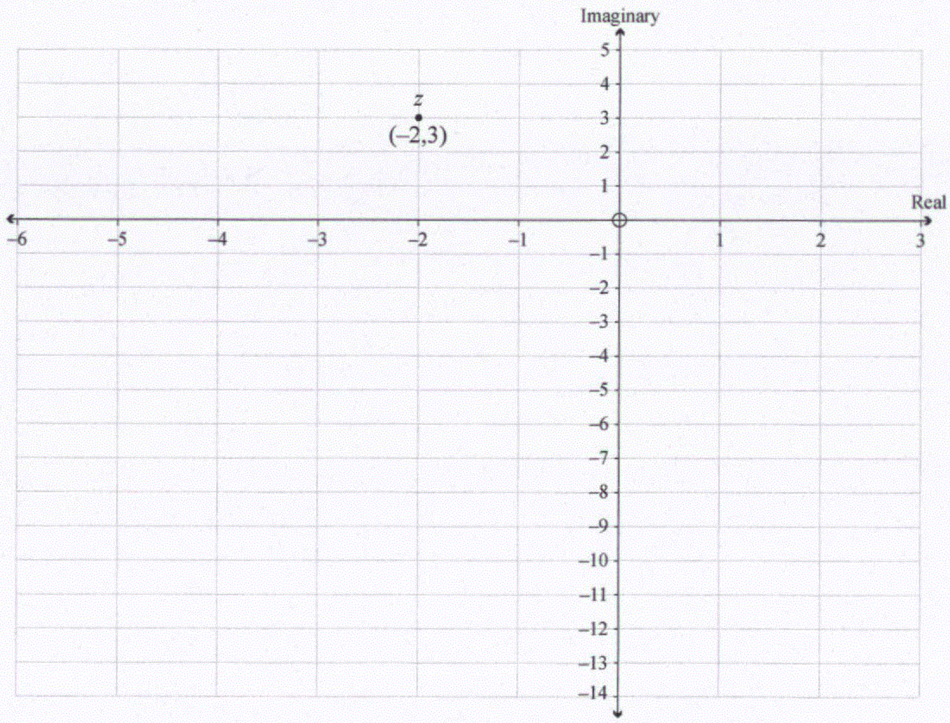
$u = 3+5i$ or $3+i$

QUESTION THREE

(a) Write $\dfrac{\sqrt{2p}}{\sqrt{2p}-\sqrt{p}}$ in the form $a+\sqrt{a}$, where p and a are real constants.

(b) In the Argand diagram below, the point (–2,3) represents the complex number z.

Show clearly, in the diagram below, the point representing $w = z^2$



Question Three continues on the next page.

(c) Find the value(s) of the real constant d, given that $z = 3 + d\mathrm{i}$ and $\bar{z} = 10dz^{-1}$.

$z = 3 + di$ $\qquad \bar{z} = \frac{10d}{z}$

$3 - di = \frac{10d}{3+di}$

$3 - di = \frac{10d(3-di)}{(3+di)(3-di)}$

$= \frac{30d - 10d^2 i}{9 - d^2 i^2}$

$= \frac{30d - 10d^2 i}{9 + d^2}$

$\frac{30d}{9+d^2} = 3$

$30d = 27 + 3d^2$

$0 = 27 - 30d + 3d^2$

$d = 9$ or $d = 1$

(d) Solve the equation $z^4 + 81k^8 = 0$, where k is a real constant.

Give your solution(s) in polar form in terms of k.

$z^4 = -81k^8$, purely real

$z^4 = 81k^8 \operatorname{cis}(\pi)$

$z_1 = 3k^2 \operatorname{cis}\left(\frac{\pi}{8}\right)$ $\qquad$ interval $= \frac{2\pi}{4} = \frac{\pi}{2}$

$z_2 = 3k^2 \operatorname{cis}\left(\frac{5}{8}\pi\right)$

$z_3 = 3k^2 \operatorname{cis}\left(-\frac{3}{8}\pi\right)$

$z_4 = 3k^2 \operatorname{cis}\left(-\frac{7}{8}\pi\right)$

(e) Given that $x+\frac{1}{x}=p$, where p is a real constant, then find the value of $x^3+\frac{1}{x^3}$,

giving your answers in terms of p.

$$x + \frac{1}{x} = p$$
$$x^2 + 1 = px$$
$$x^2 - px + 1 = 0$$

$$x = \pm \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{p \pm \sqrt{(-p)^2 - 4}}{2}$$
$$x = \frac{p \pm \sqrt{p^2 - 4}}{2} \quad , \quad = \frac{p \pm \sqrt{p^2 + 4i^2}}{2}$$
$$= \frac{p}{2} \pm \frac{\sqrt{p^2 + 4i^2}}{2}$$

$$x^3 + \frac{1}{x^3} \quad , \quad = \left(\frac{p}{2} + \frac{\sqrt{p^2+4i^2}}{2}\right)^3 + \frac{1}{x^3}$$

$$x^3 = p^3$$

$$2x = p \pm \sqrt{p^2 + 4i^2}$$
$$4x^2 = p^2 \pm p^2 + 4i^2$$

Excellence

Subject: Calculus

Standard: 91577

Total score: 21

Q	Grade score	Marker commentary
One	E8	1e identified that if *w* lay on the line $y = x$, then the real part of *w* would be equal to the imaginary. Used the discriminant to show there were no real solutions to the quadratic equation formed, t (E8) grade awarded
Two	E8	2e found the correct equation for the locus and identified the two complex numbers, *u*, which lay on that locus, t (E8) grade awarded.
Three	M5	3c used the conjugate to make the denominator of the fraction real. Then equated real parts of the equation to form and solve a quadratic and hence find the two values of *d*, r grade awarded.