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Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Level 3 Calculus 2024

91578 Apply differentiation methods in solving problems

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

Show ALL working.

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–20 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area (⁄⁄⁄). This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Achievement

TOTAL 09

QUESTION ONE

(a) Differentiate $f(x)=\sqrt{(4-9x^4)}$.

You do not need to simplify your answer.

$f(x)=(4-9\cancel{x}^4)^{1/2}$

$f'(x)=(4-9x^4)^{-1/2}.36x^3$

(b) A curve is defined by the equation $y=(x^2+3x+2)\sin x$.

Find the gradient of the tangent to this curve when $x=0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx}=\cos x(x^2+3x+2)+\sin x(2x+3)$

when $x=0$

$\frac{dy}{dx}=2+0=2$

$U=x^2+3x+2$

$U'=2x+3$

$V=\sin x$

$V'=\cos x$

(c) For the function below, find the range of values of x for which the function is decreasing.

$y=3(2x-7)^2+60\ln x+12, \ x>0$

You must use calculus and show any derivatives that you need to find when solving this problem.

$y=3(2x-7)(2x-7)+60\ln x+12$

$y=3(4x^2-14x+49)+60\ln x+12$

$y=12x^2-72x+159+60\ln x$

$\frac{dy}{dx}=24x-72+\frac{60}{x}$

when decreasing $dy/dx<0$

$24x-72+\frac{60}{x}<0$

$24x+\frac{60}{x}<72$

$24x^2+6<72x$

$24x^2-72x+60<0$

$2e^{-2x}\left(-(2x-1)-1\right) = -2x+1-1$
$= -2x$

(d) Find the x-value(s) of any stationary points on the graph of the function below, **and determine their nature**.

$y = (2x-1)e^{-2x}$

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = (2x-1)\cdot -2e^{-2x} + 2e^{-2x}$

$u = 2x-1$
$u' = 2$
$v = e^{-2x}$
$v' = -2e^{-2x}$

$= -2e^{-2x}(2x-1-1)$

$= -2e^{-2x}(2x-2)$

$u = -2e^{-2x}$
$u' = 4e^{-2x}$
$v = 2x-2$
$v' = 2$

$\frac{d^2y}{dx^2} = 2\cdot -2e^{-2x} + 4e^{-2x}(2x-2)$

$= -4e^{-2x} + 4e^{-2x}(2x-2)$

$= -4e^{-2x}(1-(2x-2))$
$= -4e^{-2x}(1-2x+2)$
$= -4e^{-2x}(-2x+3)$

when $\frac{dy}{dx} = 0$

$-2e^{-2x} = 0 \rightarrow e^{-2x} = 0$

~~which isn't possible~~, $x = 0.5$

~~or~~

$(2x-2) = 0 \quad 2x-2 = 0$
$2x = 2$
$x = 1$

$\frac{d^2y}{dx^2} = -4e^{-2x}(-2x+3)$

when $x = 0.5$

$\frac{d^2y}{dx^2} = -2.94 < 0 \therefore$ maximum

when $x = 1$

$\frac{d^2y}{dx^2} = 0.54 > 0 \therefore$ minimum

(e) A curve is defined by the equation $y = \dfrac{2x^2 - 1 - 2x\ln x}{x}$, where $x > 0$.

The curve has a point of inflection at the point P.

Find the equation of the tangent to the curve at the point P.

You must use calculus and show any derivatives that you need to find when solving this problem.

$U = 2x^2 - 1 - 2x\ln x$

$V = x$

$V' = 1$

$U' = -2 - 2\ln|x|$

$U_1 = -2x$

$U_1' = -2$

$V = \ln x$

$V' = 1/x$

$\frac{dy}{dx} = \frac{-2x}{x} + -2\ln(x)$

$= -2 - 2\ln|x|$

$\frac{dy}{dx} = \frac{x(-2-2\ln|x|) - (2x^2-1)}{x^2}$

$= \frac{-2x - 2x\ln|x| - 2x^2 + 1}{x^2}$

$\frac{d^2y}{dx^2} = \frac{x^2(\cancel{-2x} + 2\ln|x| - 4x) - 2x(-2x - 2\ln|x| - 2x^2 + 1)}{x^4}$

$\frac{d^2y}{dx^2} = 0 = \frac{2x^2\ln|x| \cancel{- 4x^3} + 4x^2 + 4x\ln|x| \cancel{+ 4x^3} - 2x}{x^4}$

$0 = \frac{2x^2\ln|x| + 4x^2 + 4x\ln|x| - 2x}{x^4}$

$V = x^2$

$V' = 2x$

$U = -2x - 2x\ln|x| - 2x^2 + 1$

$U' = -2 - (-2 - 2\ln|x|) - 4x$

QUESTION TWO

(a) A function is defined parametrically by the pair of equations:

$x = 3t^2 + 1$ and $y = \cos t$.

~~$v = 3t^2 + 1$~~

Find an expression for $\frac{dy}{dx}$.

$\frac{dx}{dt} = 6t$ $\frac{dy}{dt} = -\sin t$

$\frac{dy}{dx} = \frac{-\sin t}{6t}$ //

(b) An object is travelling in a straight line. Its displacement, in metres, is given by the formula

$s(t) = \ln(3t^2 + 5t + 2)$, where $t > 0$ and t is time, in seconds.

S V a

Find the velocity of this object when $t = 1$ second.

You must use calculus and show any derivatives that you need to find when solving this problem.

$V(t) = \frac{6t + 5}{3t^2 + 5t + 2}$

$v(t)\Big|_{t=1} = \frac{11}{10} = 1.1\ ms^{-1}$ //

$u = x^2$
$u' = 2x$
$v = \cancel{2}\cos(x^2)$
$v' = -x^2 \sin x^2$

(c) Show that $y = \sin(x^2) - \cos(x)$ is a solution to the equation

$$\frac{d^2y}{dx^2} + 4x^2y = 2\cos\left(x^2\right) + \left(1 - 4x^2\right)\cos x.$$

$$\frac{dy}{dx} = x^2 \cos x^2 + \sin x$$

$$\frac{d^2y}{dx^2} = x^2(-x^2 \sin x^2) + 2x\cos x^2 + \cos x = -x^4 \sin x^2 + 2x\cos x^2 + \cos x$$

$$4x^2y = 4x^2(\sin x^2 - \cos x) = 4x^2 \sin x^2 - 4x^2 \cos x$$

$$\frac{d^2y}{dx^2} + 4x^2y = -4x^2 \sin x^2 + 2x\cos x^2 + \cos x + 4x^2 \sin x^2 - 4x^2\cos x + \cos x$$

$$= 2\cos(x^2) + (1 - 4x^2)\cos x$$

(d) Consider the function $f(x)=\dfrac{\ln x}{x}, x>0$.

Find the coordinates of the point of inflection on the graph of the function.

You can assume that your point found is actually a point of inflection.

You must use calculus and show any derivatives that you need to find when solving this problem.

$U = \ln x$

$U' = 1/x$

$V = x$

$V' = 1$

$$f'(x) = \frac{\frac{x}{x} - \ln|x|}{x^2} = \frac{1-\ln|x|}{x^2}$$

$U = 1-\ln|x|$

$U' = -\frac{1}{x}$

$V = x^2$

$V' = 2x$

$$f''(x) = \frac{\frac{-x^2}{x} - 2x(1-\ln|x|)}{x^4}$$

$$= \frac{-x - 2x + 2x\ln|x|}{x^4}$$

$$= \frac{-3x + 2\ln|x|}{x^4}$$

$f''(x) = 0$

$$3x = \frac{2\ln|x|}{\cancel{x^4}}$$

$\Rightarrow x = 0.46$

$$f(x) = \frac{\ln|0.46|}{0.46}$$

$f(x) = -1.68$

$(0.46, -1.68)$

(e) The graph of the function $y = \frac{xe^{3x}}{2x+k}$, where k is a non-zero constant, **has a single turning point** at **Q**.

Find the x-coordinate of the point Q.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = \frac{(2x+k)(3xe^{3x}) - 2xe^{3x}}{(2x+k)^2}$

$U = xe^{3x}$

$U' = 3xe^{3x}$

$V = 2x+k$

$V' = \cancel{3}\ 2$

$= \frac{6x^2e^{3x} + 3kxe^{3x} - 2xe^{3x}}{(2x+k)^2}$

$= \frac{xe^{3x}(6x+3k-2)}{4x^2+4k+k^2}$

when $\frac{dy}{dx} = 0$

$xe^{3x} = 0$ or $6x+3k-2=0$

which isn't possible

$6x+3k-2=0$

$6x + \cancel{3k} = 2$

$3k = 2 - 6x$

$k = \frac{2}{3} - 2x$

$(2x+k)(2x+k)$

$= ax^2 + 4kx + k^2$

QUESTION THREE

(a) Differentiate $y=\sqrt{x}.\sec(6x)$.

You do not need to simplify your answer.

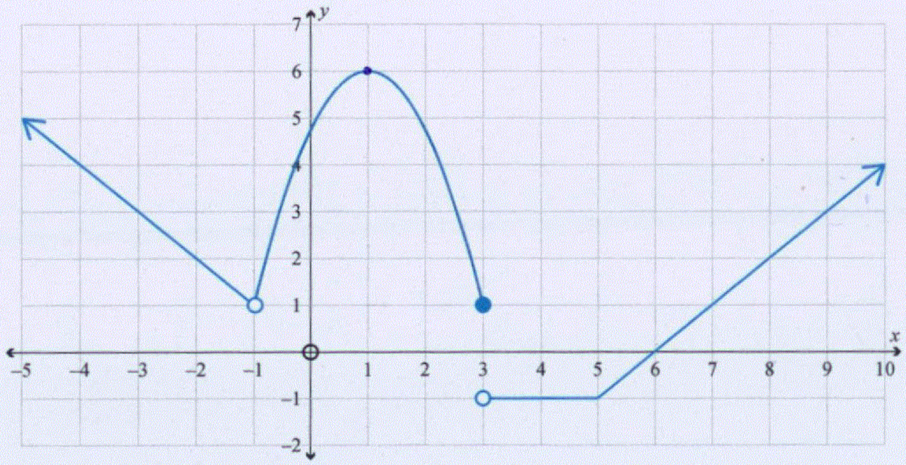
$0.5x^{-1/2}$

$U=x^{1/2}$ $U'=\frac{\cancel{x^{-1/2}}}{2}$

$V=\sec 6x$ $V'=6\sec x\tan x$

$\frac{dy}{dx}=\sqrt{x}\left(6\sec x\tan x\right)+0.5x^{-1/2}\left(\sec 6x\right)$

(b) The graph below shows the function $y=f(x)$.



(i) For the function above, find the value(s) of x where $f(x)$ is continuous but not differentiable.

$x=5,\ x=3,\ x=-1$

(ii) For the function above, find the value(s) of x where $f'(x)=0$.

$x=1,\quad x>3,\ x<5$

(iii) What is the value of $\lim\limits_{x\to -1} f(x)$?

State clearly if the value does not exist.

1

$$f(x) = \frac{(x-4)(x-1)}{(x+4)(x+1)}$$

(c) Find the x-value(s) of any stationary point(s) on the graph of the function $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 5x + 4}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

*You **do not need** to determine the nature of any stationary point(s) found.*

$$\frac{dy}{dx} = \frac{(2x-5)(x^2+5x+4) - (2x+5)(x^2-5x+4)}{(x^2+5x+4)^2}$$

$U = x^2 - 5x + 4$

$U' = 2x - 5$

$V = x^2 + 5x + 4$

$V' = 2x + 5$

$$= \frac{\cancel{2x^3} + 10x^2 + \cancel{8x} - 5x^2 - \cancel{25x} - \cancel{20} - 2x^3 + 10x^2 - 8x - 5x^2 + 25x + 20}{(x^2+5x+4)^2}$$

$$= \frac{10x^2 - 5x^2 + 10x^2 - 5x^2}{(x^2+5x+4)^2}$$

$$= \frac{10x^2}{(x^2+5x+4)^2}$$

at stationary point $dy/dx = 0$

$0 = 10x^2$

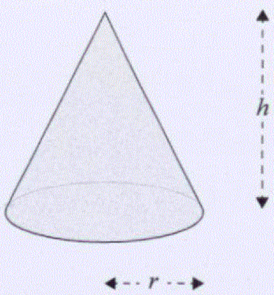
(d) Jamie is doing some baking and pouring the flour to form a conical pile.

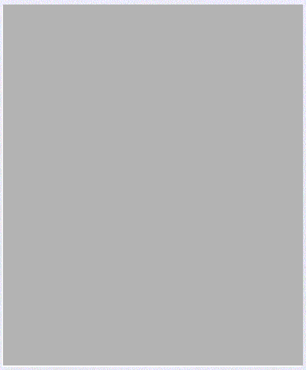
The height of the pile is always the same as the diameter of the base of the cone.

If the flour is being added at a constant rate of 3 cm³ per second, at what rate is the height increasing when the pile is 4 cm in height?

You must use calculus and show any derivatives that you need to find when solving this problem.

Jamie pouring flour





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Note that volume of a cone $= \frac{1}{3}\pi r^2 h$.

$h = 2r$

$4 = 2r$

$r = 2$

$\frac{dh}{dt} = \cancel{4}\ ?$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dh}{dr} = 2$

$\frac{dV}{dt} = 3$

$V = \frac{1}{3}\pi r^2 (4)$

$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dr} \cdot \frac{dr}{dV}$

$V = \frac{4}{3}\pi r^2$

$V = \frac{4}{3}\pi 4$

$= \frac{3.2}{11.17}$

$V = 16.755$

$= 0.537\ cms^{-1}$ (3sf)

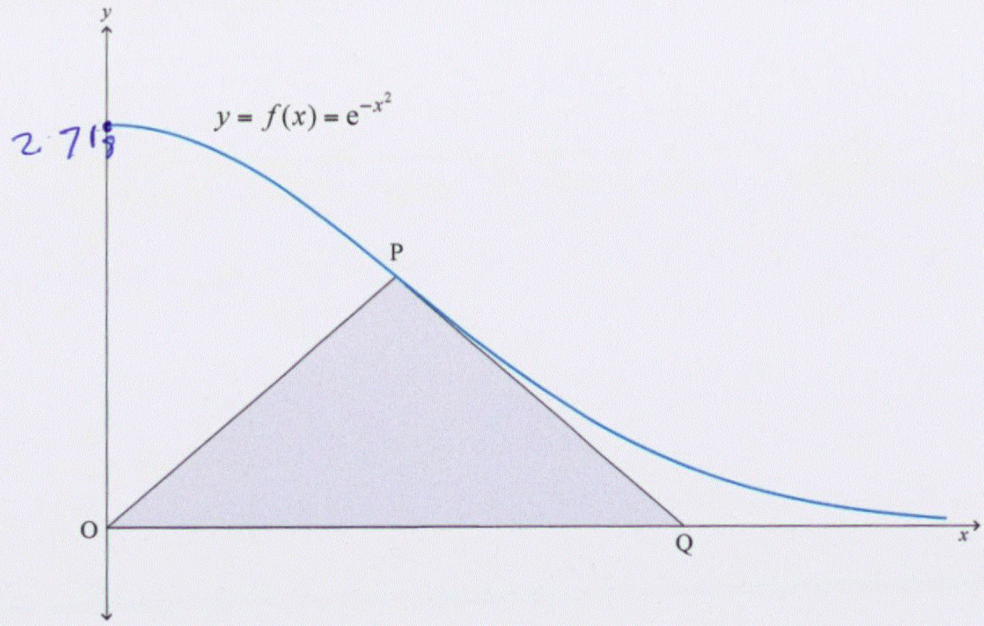
$\frac{dV}{dr} = \frac{16}{9}\pi r$

$r = 2$

$= \cancel{8}\ 11.17$

Question Three continues on the next page.

(e) The diagram below shows part of the graph of the function $f(x) = e^{-x^2}$, where $x \geq 0$.



The point P lies on the curve and the point Q lies on the x-axis so that OP = PQ, where O is the origin.

Prove that the largest possible area of the triangle OPQ is $\dfrac{1}{\sqrt{2e}}$.

You do not need to show that the area you have found is a maximum.

You must use calculus and show any derivatives that you need to find when solving this problem.

$f'(x) = -x^2 e^{-x^2}$

when $x = 0$

$f(x) = e^0 = 2.718$

Achievement

Subject: Calculus

Standard: 91578

Total score: 9

Q	Grade score	Marker commentary
One	A3	This question provides evidence for A3 because the candidate differentiated a trigonometric function involving the product rule. They also used the product rule to successfully differentiate exponential function. This candidate did not achieve an A4 because they did not differentiate a polynomial or radical function with the chain rule.
Two	A4	This question provides evidence for A4 because the candidate differentiated a variety of functions, which involved the use of the chain rule and quotient rule. The functions correctly differentiated involved parametric, trigonometric and natural log functions. This candidate did not achieve an M5 because they did not find the *y*-coordinate of the point of inflection.
Three	N2	This question provides evidence for N2 because the candidate demonstrated an ability to differentiate a polynomial function using the quotient rule. This candidate did not get A3 as they did not apply the product rule to differentiate a trigonometric function. They also did not identify the features of a piecewise function.