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Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Level 3 Calculus 2024

91578 Apply differentiation methods in solving problems

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

Show ALL working.

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–20 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 21

QUESTION ONE

(a) Differentiate $f(x)=\sqrt{(4-9x^4)}$.

You do not need to simplify your answer.

$f(x) = (4-9x^4)^{\frac{1}{2}}$ $\quad f'(x) = \frac{1}{2}x - 36x^3(4-9x^4)^{-\frac{1}{2}}$

(b) A curve is defined by the equation $y=(x^2+3x+2)\sin x$.

Find the gradient of the tangent to this curve when $x=0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = (2x+3)\sin x + (x^2+3x+2)\cos x$

$x=0$: $\frac{dy}{dx} = (2(0)+3)\sin(0) + ((0)^2+3(0)+2)\cos(0)$

$= 2$

(c) For the function below, find the range of values of x for which the function is decreasing.

$y=3(2x-7)^2+60\ln x+12,\ x>0$

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = 12(2x-7) + \frac{60}{x}$

$= 24x - 84 + \frac{60}{x}$

function decreasing: $\frac{dy}{dx} < 0$

$24x - 84 + \frac{60}{x} < 0$

$24x^2 - 84x + 60 < 0$

$2x^2 - 7x + 5 < 0$

$(2x-5)(x-1) < 0$

$1 < x < \frac{5}{2}$

12

$x^2 - 28x + 49$

$12x$

(d) Find the x-value(s) of any stationary points on the graph of the function below, **and determine their nature**.

$y = (2x - 1)e^{-2x}$

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = 2e^{-2x} - 2(2x-1)e^{-2x}$

$= 2e^{-2x} - 4xe^{-2x} + 2e^{-2x}$

$= 4e^{-2x} - 4xe^{-2x}$

stationary point : $\frac{dy}{dx} = 0$

$0 = 4e^{-2x} - 4xe^{-2x}$

$4xe^{-2x} = 4e^{-2x}$

$x = 1$

$\frac{d^2y}{dx^2} = -8e^{-2x} - 4e^{-2x} + 8xe^{-2x}$

$= -12e^{-2x} + 8xe^{-2x}$

$x = 1$: $\frac{d^2y}{dx^2} = -12e^{-2(1)} + 8(1)e^{-2(1)}$

$= -12e^{-2} + 8e^{-2}$

$= -4e^{-2}$

$= -0.541$

negative 2nd derivative value so maximum point.

(e) A curve is defined by the equation $y = \frac{2x^2 - 1 - 2x\ln x}{x}$, where $x > 0$.

The curve has a point of inflection at the point P.

Find the equation of the tangent to the curve at the point P.

You must use calculus and show any derivatives that you need to find when solving this problem.

$y = 2x - \frac{1}{x} - \cancel{\frac{2x\ln x}{x}}\ 2\ln x$

$\frac{dy}{dx} = 2 + x^{-2} - \frac{2}{x}$

$= 2 + \frac{1}{x^2} - \frac{2}{x}$

$\frac{d^2y}{dx^2} = -2x^{-3} + 2x^{-2}$

$= -\frac{2}{x^3} + \frac{2}{x^2}$

point of inflection : $\frac{d^2y}{dx^2} = 0$

$0 = -\frac{2}{x^3} + \frac{2}{x^2}$

$\frac{2}{x^3} = \frac{2}{x^2}$

$x^3 = x^2$

$x = 1$ $\quad y = \frac{2(1)^2 - 1 - 2(1)\ln(1)}{1}$

$= 1$

P: $(1, 1)$ $\quad x = 1$: $\frac{dy}{dx} = 2 + \frac{1}{(1)^2} - \frac{2}{(1)}$

$= 1$

$1 = 1(1) + c$

$c = 0$

$y = x$

QUESTION TWO

(a) A function is defined parametrically by the pair of equations:

$x = 3t^2 + 1$ and $y = \cos t$.

Find an expression for $\frac{dy}{dx}$.

$dx = 6t \ dt \quad dy = -\sin t \ dt$

$\frac{dt}{dx} = \frac{1}{6t} \quad \frac{dy}{dt} = -\sin t$

$\frac{dy}{dx} = \frac{1}{6t} \cdot -\sin t$

$= -\frac{1}{6t}\sin t$

(b) An object is travelling in a straight line. Its displacement, in metres, is given by the formula

$s(t) = \ln(3t^2 + 5t + 2)$, where $t > 0$ and t is time, in seconds.

Find the velocity of this object when $t = 1$ second.

You must use calculus and show any derivatives that you need to find when solving this problem.

$v(t) = \frac{6t+5}{3t^2+5t+2}$

$v(1) = \frac{6(1)+5}{3(1)^2+5(1)+2}$

$= \frac{11}{10}$

$= 1.1 \ ms^{-1}$

(c) Show that $y = \sin(x^2) - \cos(x)$ is a solution to the equation

$$\frac{d^2y}{dx^2} + 4x^2y = 2\cos\left(x^2\right) + \left(1 - 4x^2\right)\cos x.$$

$$\frac{dy}{dx} = 2x\cos x^2 + \sin x$$

$$\frac{d^2y}{dx^2} = 2\cos x^2 - \cancel{2x}\,4x^2 \sin x^2 + \cos x$$

$$= \cancel{3\cos x - 2x\sin x}$$

$$\cancel{LHS = 3\cos x - 2x\sin x + 4x^2(\sin x^2 - \cos x)}$$

$$\cancel{= 3\cos x - 2x\sin x + 4x\sin x^2 - 4\cos x^2 \cos x}$$

$$LHS = 2\cos x^2 - 4x^2\sin x^2 + \cos x + 4x^2(\sin x^2 - \cos x)$$

$$= 2\cos x^2 - 4x^2\sin x^2 + \cos x + 4x^2\sin x^2 - 4x^2\cos x$$

$$= 2\cos x^2 + \cos x - 4x^2\cos x$$

$$= 2\cos(x^2) + (1 - 4x^2)\cos x$$

$$RHS = 2\cos(x^2) + (1 - 4x^2)\cos x$$

LHS = RHS so $y = \sin(x^2) - \cos(x)$ is a solution.

(d) Consider the function $f(x) = \frac{\ln x}{x}, x > 0$.

Find the coordinates of the point of inflection on the graph of the function.

You can assume that your point found is actually a point of inflection.

You must use calculus and show any derivatives that you need to find when solving this problem.

$f'(x) = \frac{\frac{x}{x} - \ln x}{x^2}$

$= \frac{1 - \ln x}{x^2}$

$f''(x) = \frac{-\frac{x^2}{x} - 2x(1 - \ln x)}{(x^2)^2}$

$= \frac{-x - 2x + 2x\ln x}{x^4}$

$= \frac{-3x + 2x\ln x}{x^4}$

$= \frac{-3 + 2\ln x}{x^3}$

point of inflecton: $f''(x) = 0$

$0 = \frac{-3 + 2\ln x}{x^3}$

$0 = -3 + 2\ln x$

$3 = 2\ln x$

$\frac{3}{2} = \ln x$

$x = e^{\frac{3}{2}} = 4.4817$

(e) The graph of the function $y = \dfrac{xe^{3x}}{2x+k}$, where k is a non-zero constant, **has a single turning point** at **Q**.

Find the x-coordinate of the point Q.

You must use calculus and show any derivatives that you need to find when solving this problem.

$$\frac{dy}{dx} = \frac{(2x+k)(3xe^{3x}+e^{3x}) - 2(xe^{3x})}{(2x+k)^2}$$

$$= \frac{6x^2e^{3x}+2xe^{3x}+3kxe^{3x}+ke^{3x}-2xe^{3x}}{(2x+k)^2}$$

$$= \frac{e^{3x}(6x^2+3kx+k)}{(2x+k)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(2x+k)^2((12x+3k)e^{3x}+3e^{3x}(6x^2+3kx+k)) - (8x+4k)(e^{3x}(6x^2+3kx+k))}{(2x+k)^4}$$

$$= (2x+k)^2(18x^2e^{3x}+12xe^{3x}+9kxe^{3x}+6ke^{3x}) -$$

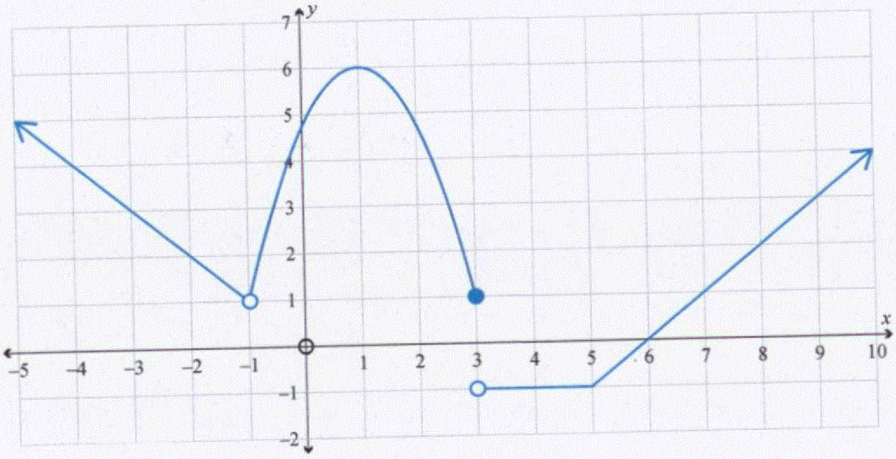
QUESTION THREE

(a) Differentiate $y=\sqrt{x}.\sec(6x)$.

You do not need to simplify your answer.

$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}\sec 6x+\sqrt{x}\times 6\sec 6x\tan 6x$

(b) The graph below shows the function $y=f(x)$.



(i) For the function above, find the value(s) of x where $f(x)$ is continuous but not differentiable.

$x=5$

(ii) For the function above, find the value(s) of x where $f'(x)=0$.

$x=1$, $x>5$

(iii) What is the value of $\lim_{x\to -1} f(x)$?

State clearly if the value does not exist.

$\lim_{x\to -1} f(x)=1$

(c) Find the x-value(s) of any stationary point(s) on the graph of the function $f(x)=\dfrac{x^2-5x+4}{x^2+5x+4}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

You ***do not need*** *to determine the nature of any stationary point(s) found.*

$$f'(x)=\frac{(x^2+5x+4)(2x-5)-(x^2-5x+4)\cancel{(x^2+5)}(2x+5)}{(x^2+5x+4)^2}$$

$$\cancel{\frac{(2x^3+10x^2-5x^2+8x-25x-20)-(x^4-5x^3+4x^2+5x^2-25x+20)}{(x^2+5x+4)^2}}$$

$$\cancel{=\frac{-x^4+7x^3-4x^2+8x-40}{(x^2+5x+4)^2}}$$

$$=\frac{(2x^3+10x^2-5x^2+8x-25x-20)-(2x^3-10x^2+5x^2+8x-25x+20)}{(x^2+5x+4)^2}$$

$$=\frac{(2x^3+5x^2-17x-20)-(2x^3-5x^2-17x+20)}{(x^2+5x+4)^2}$$

$$=\frac{10x^2-40}{(x^2+5x+4)^2}$$

stationary point: $\frac{dy}{dx}=0$

$$0=\frac{10x^2-40}{(x^2+5x+4)^2}$$

$$0=10x^2-40$$

$$10x^2=40$$

$$x^2=4$$

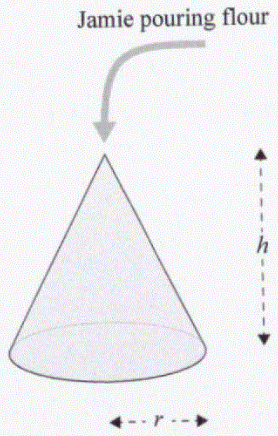
$$x=\pm 2$$

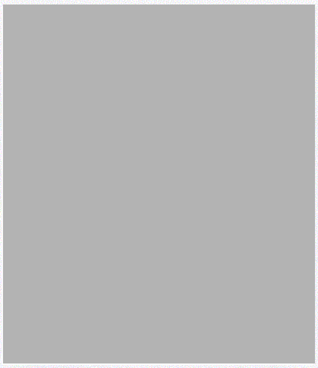
(d) Jamie is doing some baking and pouring the flour to form a conical pile.

The height of the pile is always the same as the diameter of the base of the cone.

If the flour is being added at a constant rate of 3 cm^3 per second, at what rate is the height increasing when the pile is 4 cm in height?

You must use calculus and show any derivatives that you need to find when solving this problem.





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Note that volume of a cone $= \frac{1}{3}\pi r^2 h$.

$h = 2r \qquad \frac{dV}{dt} = 3 \qquad$ find $\frac{dh}{dt}$

$r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

$= \frac{1}{12}\pi h^3$

$dV = \frac{1}{4}\pi h^2 \, dh$

$\frac{dh}{dV} = \frac{4}{\pi h^2}$

$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV}$

$= 3 \times \frac{4}{\pi h^2}$

$= \frac{12}{\pi h^2}$

$h = 4: \quad \frac{dh}{dt} = \frac{12}{\pi (4)^2}$

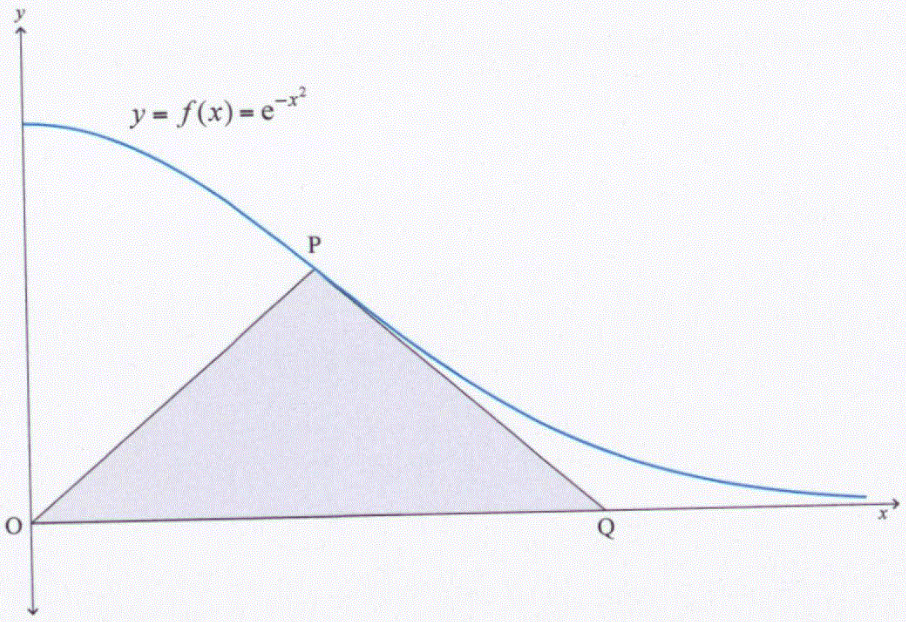
$= \frac{12}{16\pi}$

$= \frac{3}{4\pi}$

$= 0.2387 \text{ cm s}^{-1}$

Question Three continues on the next page.

(e) The diagram below shows part of the graph of the function $f(x) = e^{-x^2}$, where $x \geq 0$.



The point P lies on the curve and the point Q lies on the x-axis so that OP = PQ, where O is the origin.

Prove that the largest possible area of the triangle OPQ is $\frac{1}{\sqrt{2e}}$.

You do not need to show that the area you have found is a maximum.

You must use calculus and show any derivatives that you need to find when solving this problem.

$A = xy \qquad y = e^{-x^2}$

$A = xe^{-x^2}$

$A' = e^{-x^2} - 2x^2 e^{-x^2}$

$= e^{-x^2}(1 - 2x^2)$

maximum area: $A' = 0$

$0 = e^{-x^2}(1 - 2x^2)$

$e^{-x^2} \neq 0$

as ln |0| does not exist

$1 - 2x^2 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

$x \geq 0$

$\therefore \quad x = \frac{1}{\sqrt{2}}$

maximum area when $x = \frac{1}{\sqrt{2}}$

$A = \frac{1}{\sqrt{2}} e^{-\left(\frac{1}{\sqrt{2}}\right)^2}$

$= \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{2}\sqrt{e}}$

$= \frac{1}{\sqrt{2e}}$

Excellence

Subject: Calculus

Standard: 91578

Total score: 21

Q	Grade score	Marker commentary
One	E8	This candidate simplified to find the first derivative of the function given in the excellence question. They then found the second derivative and solved equal to zero to find the x-coordinate of the point of inflection. They then found the equation of the tangent at this point. By completing the excellence question successfully they gained the maximum E8 for this question. This candidate also successfully solved all the achievement and merit parts to this question.
Two	M5	This question provides evidence for an M5 because the candidate found the correct first and second derivative of the trigonometric function in part c. They then proved that the function given was a solution to the differential equation. This candidate did not achieve higher than the M5 code for this question because they did not fully complete the problem in part d by finding the y-coordinate of point of inflection for the function given. For the excellence question the candidate differentiated the given function with the product and quotient rule but did not make any further progress towards finding the value of the pronumeral k that was required for an r grade.
Three	E8	For this question, the candidate established a correct model for the area of the full triangle. They then differentiated this and solved when equal to 0. In doing this, the candidate clearly communicated that the exponential component could not equal zero when solving to find the solution for x. They also acknowledged that the negative solution for x was not possible in the context of the problem. They then able proved that the maximum area of the triangle could be given by $\frac{1}{\sqrt{2e}}$. By completing the excellence question successfully, they gained the maximum E8 for this question. This candidate also successfully solved nearly all of the achievement and merit levels problems of this question, only making one mistake when identifying the features of the piecewise function in part b.