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Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Level 3 Calculus 2024

91578 Apply differentiation methods in solving problems

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply differentiation methods in solving problems.	Apply differentiation methods, using relational thinking, in solving problems.	Apply differentiation methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

Show ALL working.

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–20 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Merit

TOTAL 15

QUESTION ONE

(a) Differentiate $f(x) = \sqrt{(4 - 9x^4)}$.

You do not need to simplify your answer.

$f'(x) = \dfrac{1}{2\sqrt{4-9x^4}} \times -36x^3$

$= \dfrac{-18x^3}{\sqrt{4-9x^4}}$

(b) A curve is defined by the equation $y = (x^2 + 3x + 2)\sin x$.

Find the gradient of the tangent to this curve when $x = 0$.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = (2x+3)\sin x + (x^2+3x+2)\cos x$

$\frac{dy}{dx} = 3\sin 0 + (0^2 + 3\times 0 + 2)\cos 0$

$\frac{dy}{dx} = 2$

(c) For the function below, find the range of values of x for which the function is decreasing.

$y = 3(2x-7)^2 + 60\ln x + 12, \ x > 0$

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = 6(2x-7)\cdot 2 + \frac{60}{x}$

$\frac{dy}{dx} = 12(2x-7) + \frac{60}{x}$

$12(2x-7) + \frac{60}{x} < 0$

$24x - 84 < \frac{60}{x} \times x$

$24x^2 - 84x < 60$

$24x^2 - 84x - 60 < 0$

$12(2x^2 - \cancel{84x}\ 7x - 5) < 0$

$2x^2 - 7x - 5 < 0$

$\cancel{x = \frac{7+\sqrt{89}}{}}$ $x = \pm 4.108$

$-4.108 < x < 4.108$

(d) Find the x-value(s) of any stationary points on the graph of the function below, **and determine their nature**.

$$y = (2x-1)e^{-2x}$$

You must use calculus and show any derivatives that you need to find when solving this problem.

$\frac{dy}{dx} = 2e^{-2x} + (2x-1)e^{-2x} \cdot -2$

$\frac{dy}{dx} = 2e^{-2x} - 2(2x-1)e^{-2x}$

Stationary points: $\frac{dy}{dx} = 0$

$0 = 2e^{-2x} - 2(2x-1)e^{-2x}$

$0 = 2e^{-2x} - (4x+2)e^{-2x}$

$0 = e^{-2x}(2-4x+2)$

$0 = 2-4x+2$

$0 = 4-4x$

$\frac{4x}{4} = \frac{4}{4}$

$x = 1$

$\cancel{\frac{d^2y}{dx^2}}$ $\frac{dy}{dx} = e^{-2x}(\cancel{4x+2}\ 4-4x)$

$\frac{d^2y}{dx^2} = -2e^{-2x}(4-4x) - 4e^{-2x}$

when $x=1$,

$\frac{d^2y}{dx^2} = -2e^{-2\times1}(4-4) - 4e^{-2\times1}$

$= -0.54$ therefore as $\frac{d^2y}{dx^2}$ is a negative number, this stationary point $(x=1)$ is a maximum.

(e) A curve is defined by the equation $y = \dfrac{2x^2 - 1 - 2x\ln x}{x}$, where $x > 0$.

The curve has a point of inflection at the point P.

Find the equation of the tangent to the curve at the point P.

You must use calculus and show any derivatives that you need to find when solving this problem.

$\cancel{y = \frac{2x^2}{x} - \frac{1}{x}}$

$$\frac{dy}{dx} = \frac{x(4x-2) - (2x^2 - 1 - 2x\ln x)}{x^2}$$

$$\frac{dy}{dx} = \frac{4x^2 - 2x - 2x^2 + 1 + 2x\ln x}{x^2}$$

$$0 = \frac{2x^2 - 2x + 1 + 2x\ln x}{x^2}$$

$$0 = 2x^2 - 2x + 1 + 2x\ln x$$

QUESTION TWO

(a) A function is defined parametrically by the pair of equations:

$x = 3t^2 + 1$ and $y = \cos t$.

Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$\frac{dx}{dt} = 6t$

$\frac{dt}{dx} = \frac{1}{6t}$

$\frac{dy}{dt} = -\sin t$

$\frac{dy}{dx} = \frac{-\sin t}{6t}$

(b) An object is travelling in a straight line. Its displacement, in metres, is given by the formula

$s(t) = \ln(3t^2 + 5t + 2)$, where $t > 0$ and t is time, in seconds.

Find the velocity of this object when $t = 1$ second.

You must use calculus and show any derivatives that you need to find when solving this problem.

$v(t) = \frac{1}{3t^2+5t+2} \times 6t+5$

$v(t) = \frac{6t+5}{3t^2+5t+2}$

$v(t) = \frac{6(1)+5}{3(1)^2+5(1)+2}$

$v(1) = 1.1\ ms^{-1}$

(c) Show that $y = \sin(x^2) - \cos(x)$ is a solution to the equation

$$\frac{d^2y}{dx^2} + 4x^2y = 2\cos\left(x^2\right) + \left(1 - 4x^2\right)\cos x.$$

$\frac{dy}{dx} = 2x\cos(x^2) + \sin(x)$

$\frac{d^2y}{dx^2} = -4x^2\sin(x^2) + \cos(x)$

~~cos y =~~

$\frac{d^2y}{dx^2} + 4x^2y = -4x^2\sin(x^2) + \cos(x) + 4x^2(\sin(x^2) - \cos(x))$

$= -4x^2\sin(x^2) + \cos(x) + 4x^2\sin(x^2) - 4x^2\cos(x)$

~~sin~~ $\cos x - 4x^2\cos x$

$= (1 - 4x^2)\cos x$

(d) Consider the function $f(x)=\dfrac{\ln x}{x}, x>0$.

Find the coordinates of the point of inflection on the graph of the function.

You can assume that your point found is actually a point of inflection.

You must use calculus and show any derivatives that you need to find when solving this problem.

$$f'(x)=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}$$

$$f'(x)=\frac{1-\ln x}{x^2}$$

$$0=\frac{1-\ln x}{x^2}$$

$$0=1-\ln x$$

$$e^{\ln x}=e^{1}$$

$$x=e^1$$

$$x=2.72$$

$$y=\frac{\ln(2.72)}{2.72}$$

$$y=0.368$$

(e) The graph of the function $y = \frac{xe^{3x}}{2x+k}$, where k is a non-zero constant, **has a single turning point** at **Q**.

Find the x-coordinate of the point Q.

You must use calculus and show any derivatives that you need to find when solving this problem.

$$\frac{dy}{dx} = \frac{xe^{3x}.3(2x+k) - 2(xe^{3x})}{(2x+k)^2}$$

$$\times(2x+k)^2 \quad 0 = \frac{3xe^{3x}(2x+k) - 2(xe^{3x})}{(2x+k)^2} \quad \times(2x+k)^2$$

$$0 = 3xe^{3x}(2x+k) - 2xe^{3x}$$

$$\frac{0}{e^{3x}} = \frac{e^{3x}(6x^2 + 3kx - 2x)}{e^{3x}}$$

$$0 = 6x^2 + 3kx - 2x$$

$$0 = x(6x + 3k - 2)$$

$$x = 0 \text{ , or } 6x + 3k - 2 = 0$$

$$6x + 3k = 2$$

$$\cancel{x = 2 - 3k}$$

$$k = \frac{2}{3} - 2x$$

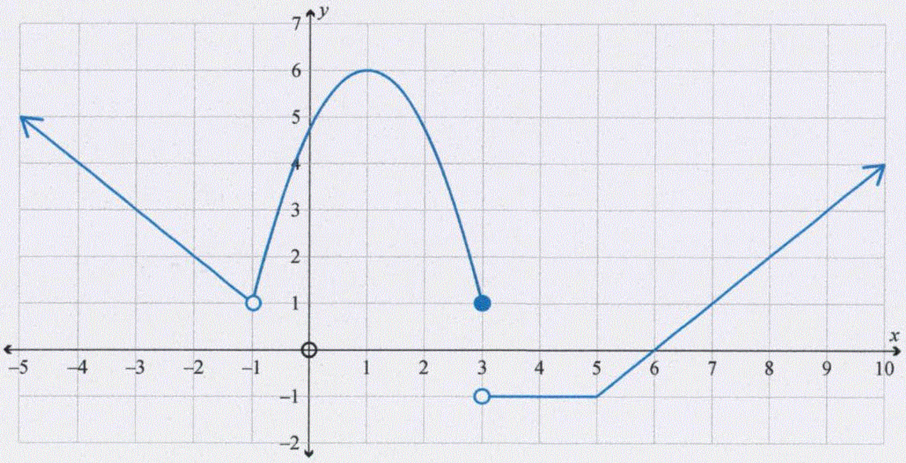
QUESTION THREE

(a) Differentiate $y = \sqrt{x}.\sec(6x)$.

You do not need to simplify your answer.

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cdot \sec(6x) + \cancel{se}\ 6\sec 6x \tan 6x \cdot \sqrt{x}$

(b) The graph below shows the function $y = f(x)$.



(i) For the function above, find the value(s) of x where $f(x)$ is continuous but not differentiable.

~~[illegible]~~, $x = 5$

(ii) For the function above, find the value(s) of x where $f'(x) = 0$.

~~x~~ $3 < x < 5$, $x = 1$

(iii) What is the value of $\lim\limits_{x \to -1} f(x)$?

State clearly if the value does not exist.

1

(c) Find the x-value(s) of any stationary point(s) on the graph of the function $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 5x + 4}$.

You must use calculus and show any derivatives that you need to find when solving this problem.

*You **do not need** to determine the nature of any stationary point(s) found.*

$$f'(x) = \frac{(2x-5)(x^2+5x+4) - (2x+5)(x^2-5x+4)}{(x^2+5x+4)^2}$$

$$\times(x^2+5x+4)^2 \quad 0 = \frac{(2x-5)(x^2+5x+4) - (2x+5)(x^2-5x+4)}{(x^2+5x+4)^2} \times (x^2+5x+4)^2$$

$$0 = (2x-5)(x^2+5x+4) - (2x+5)(x^2-5x+4)$$

$$0 = 2x^3 + 10x^2 + 8x - 5x^2 - 25x - 20 - (2x^3 - 10x^2 + 8x + 5x^2 - 25x + 20)$$

$$0 = 2x^3 + 5x^2 - 17x - 20 - (2x^3 - 5x^2 - 17x + 20)$$

$$0 = \cancel{2x^3} + 5x^2 - \cancel{17x} - 20 - \cancel{2x^3} + 5x^2 + \cancel{17x} - 20$$

$$0 = 10x^2 - 40$$

$$\frac{0}{10} = \frac{10(x^2 - 4)}{10}$$

$$0 = x^2 - 4$$

$$x = 2, \quad x = -2$$

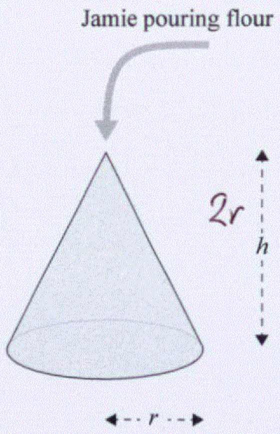
(d) Jamie is doing some baking and pouring the flour to form a conical pile.

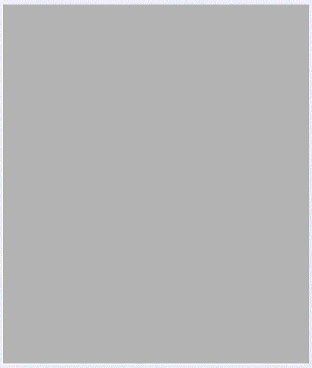
The height of the pile is always the same as the diameter of the base of the cone.

If the flour is being added at a constant rate of 3 cm³ per second, at what rate is the height increasing when the pile is 4 cm in height?

You must use calculus and show any derivatives that you need to find when solving this problem.

Note that volume of a cone $= \frac{1}{3}\pi r^2 h$.





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$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{h}{2} = \frac{2r}{2}$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi (0.5h)^2 \cdot h$$

$$V = \frac{1}{3}\pi (0.25h^3)$$

$$V = \frac{0.25h^3 \cdot \pi}{3}$$

$$\frac{dV}{dh} = \frac{0.75\pi h^2}{3}$$

$$= 0.25\pi h^2$$

$$\frac{dh}{dV} = \frac{1}{0.25\pi h^2}$$

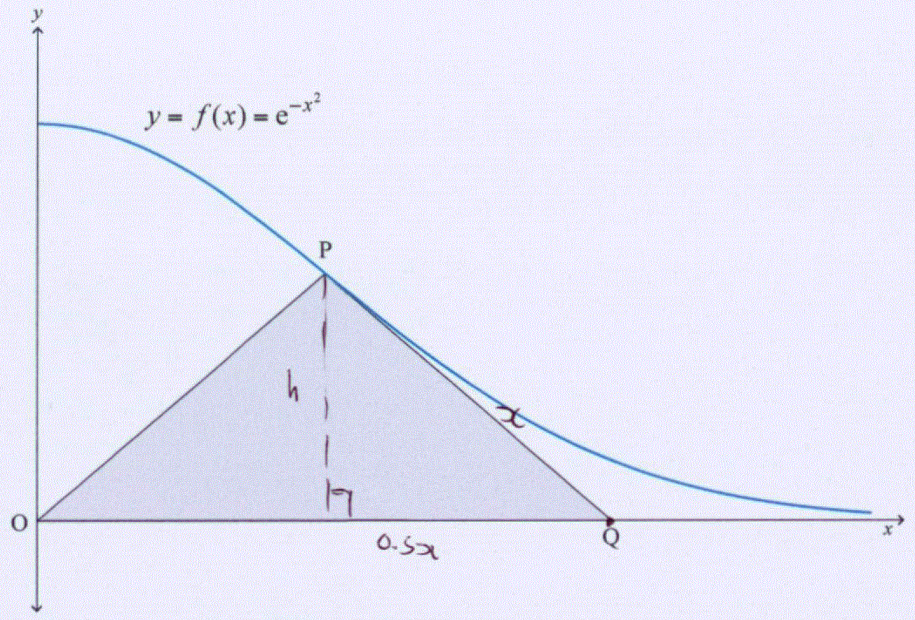
$$\frac{dh}{dt} = \frac{1}{0.25\pi h^2} \times 3$$

$$= \frac{3}{0.25\pi (4)^2}$$

$$= 0.239 \text{ cm/s}$$

Question Three continues
on the next page.

(e) The diagram below shows part of the graph of the function $f(x) = e^{-x^2}$, where $x \geq 0$.



The point P lies on the curve and the point Q lies on the x-axis so that OP = PQ, where O is the origin.

Prove that the largest possible area of the triangle OPQ is $\frac{1}{\sqrt{2e}}$.

You do not need to show that the area you have found is a maximum.

You must use calculus and show any derivatives that you need to find when solving this problem.

$A = \frac{1}{2}bh$

$b = x \quad h = e^{-x^2}$

$A = \frac{xe^{-x^2}}{2}$

$\frac{dA}{dx} = \frac{-2x \cdot xe^{-x^2}}{2}$

$= \frac{-2x^2e^{-x^2}}{2}$

$\frac{dA}{dx} = -x^2e^{-x^2}$

$0 = \frac{-x^2}{e^{x^2}}$

$h = \sqrt{x^2 + (0.5x)^2}$

$h = \sqrt{x^2 + 0.25x^2}$

$h = \sqrt{1.25x^2}$

~~$A =$~~ $A = \frac{bh}{2}$

$A = \cancel{\sqrt{x^2 + 0.25x^2}}$

$A = \frac{\sqrt{1.25x^2} \cdot x}{2}$

~~$A = x \cdot (1.25x) \cdot x$~~

$A = \frac{x(1.25x^2)^{0.5}}{2}$

$\frac{dA}{dx} = \frac{(1.25x^2)^{0.5} + 0.5x(1.25x^2)^{-0.5} \cdot 2.5x}{2}$

$$\cancel{\frac{dA}{dx} = (1.25x^2)^{0.5} + 1.25x^2}$$

$$A = \frac{bh}{2}$$

$$\cancel{A = \frac{x}{2}} \quad A = \frac{x \cdot e^{-x^2}}{2}$$

$$A = \frac{x}{e^{x^2}}$$

$$\frac{dA}{dx} = \frac{e^{x^2} - 2xe^{x^2}}{(e^{x^2})^2}$$

$$0 = \frac{e^{x^2} - 2xe^{x^2}}{e^{2x^2}}$$

$$0 = e^{x^2} - 2xe^{x^2}$$

$$2xe^{x^2} = e^{x^2}$$

Merit

Subject: Calculus

Standard: 91578

Total score: 15

Q	Grade score	Marker commentary
One	M5	This question provides evidence for an M5 because the candidate found the correct x-coordinate of the stationary point on the exponential graph and used the second derivative to determine its nature. This candidate did not achieve higher than the M5 code for this question because did not apply algebraic skills to solve a quadratic inequation to find the correct interval where a function is decreasing. They also did not apply the quotient rule to find the x coordinate of the point of inflection for the function in Q1e.
Two	A4	This question provides evidence for an A4 because both of the achievement level questions parts a and b were successfully completed and the correct first derivatives found for the functions in the merit questions. For part c, the candidate did not find the correct second derivative that was needed to prove that $y = \sin(x^2) - \cos(x)$ was a solution to the differential equation. For part d, they used the quotient rule correctly to find the derivative of the given function. They then incorrectly set this equal to zero, rather than the second derivative and setting that equal to zero to find the coordinates of the point of inflection. The candidate did not apply both the product rule and quotient rule to find the correct derivative in the excellence question.
Three	M6	This question provides evidence for an M6 because the candidate has successfully completed two of the r questions. This candidate applied the quotient rule to find the x-coordinates of the stationary points for the function in part c, clearly stating that the derivative is 0 at these stationary points. They also correctly solved the related rates of change problem in part d involving the volume of the flour. For the excellence question, they successfully established the area function, but did not differentiate it, solve and therefore prove the maximum area of the triangle.