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Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Level 3 Calculus 2024

91579 Apply integration methods in solving problems

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply integration methods in solving problems.	Apply integration methods, using relational thinking, in solving problems.	Apply integration methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

Show ALL working.

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Excellence

TOTAL 21

QUESTION ONE

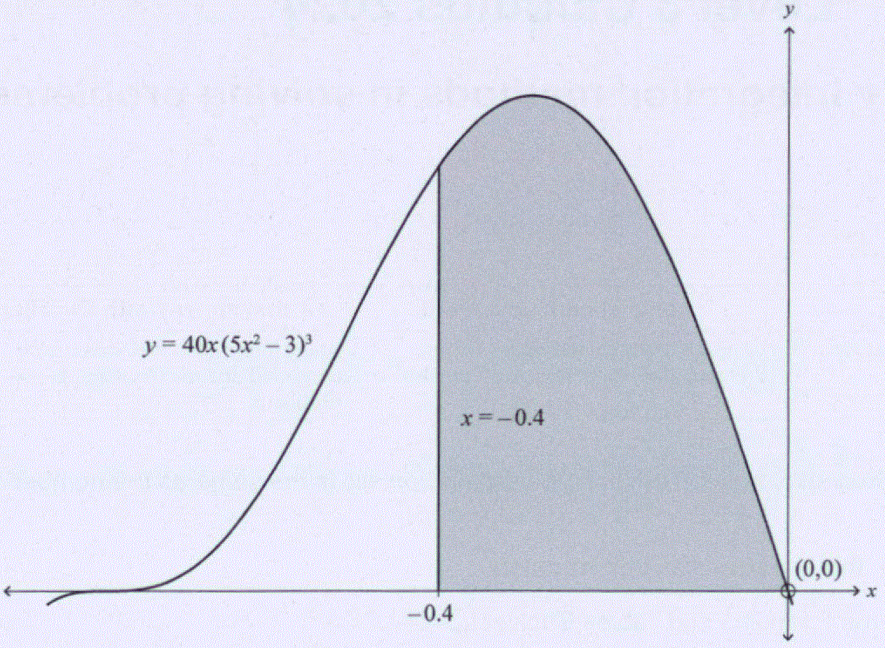
(a) Find $\int 6\sec(2x)\tan(2x)\,\mathrm{d}x$.

$3\sec(2x) + c$

(b) The graph below shows the function $y = 40x(5x^2 - 3)^3$.

Find the shaded area.

You must use calculus and show the results of any integration needed to solve the problem.



(c) An object's velocity can be modelled by the equation $v(t) = 26.4\sqrt[3]{t}$, $\rightarrow$ $26.4t^{1/3}$

where v is the velocity of the object in m s^{-1},
and t is the time in seconds since the start of timing.

Initially, the object was 360 metres from a point P.

Calculate the distance that this object has travelled from the point P when it has reached a velocity of 264 m s^{-1} .

You must use calculus and show the results of any integration needed to solve the problem.

$V'(t) = 19.8t^{4/3} + c$

$d(0) = 360$ ~~$19.8t^{4/3} + c = 360$~~

$19.8(0)^{4/3} + c = 360$ $c = 360$

$d(t) = 19.8t^{4/3} + 360$

when $V = 264 \text{ ms}^{-1}$ $t =$

$264 = 26.4t^{1/3}$ $t^{1/3} = 10$ $t = 1000$

$d(1000) = 19.8(1000)^{4/3} + 360$

$d(1000) = 198360\text{m}$

$\boxed{\text{19360m From point p}}$

(d) Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 24\cos(3x)\sin(x)$.

Given that $y = 6$ when $x = \frac{\pi}{3}$, find the value(s) of y when $x = \frac{\pi}{2}$.

You must use calculus and show the results of any integration needed to solve the problem.

$$\frac{dy}{dx} = 12\sin(4x) + 12\sin(2x)$$

$$y = -3\cos(4x) - 6\cos(2x) + c$$

$$6 = -3\cos\left(\underset{-1/2}{4 \times \frac{\pi}{3}}\right) - 6\cos\left(\underset{-1/2}{2 \times \frac{\pi}{3}}\right) + c$$

$$c = \frac{21}{2} \text{ or } 10.5$$

$$y = -3\cos(4x) - 6\cos(2x) + 10.5$$

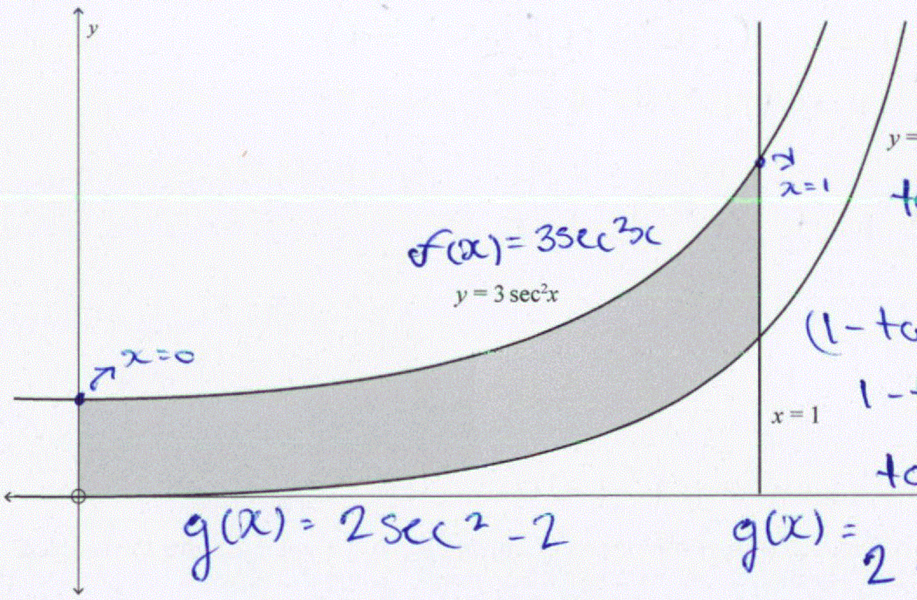
at $x = \frac{\pi}{2}$ $y =$

$$y = -3\cos\left(4 \times \frac{\pi}{2}\right) - 6\cos\left(2 \times \frac{\pi}{2}\right) + 10.5$$

$$y = -3 - 6 + 10.5$$

$$\boxed{y = 4.5}$$

(e) The graph below shows the curves $y = 3\sec^2 x$ and $y = 2\tan^2 x$.



$f(x) = 3\sec^2 x$

$x=0$ $\quad x=1$

$g(x) = 2\sec^2 - 2$ $\qquad g(x) = 2 - \frac{4\tan x}{\tan 2x}$

$g(x) = 2\tan^2 x$

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$(1-\tan^2 x)(\tan 2x) = 2\tan$

$1 - \tan^2 x = \frac{2\tan x}{\tan 2x}$

$\tan^2 x = 1 - \frac{2\tan x}{\tan 2x}$

Find the area of the shaded region enclosed by the two curves, $x = 1$, and the y-axis.

You must use calculus and show the results of any integration needed to solve the problem.

$\int f(x) = 3\tan x \qquad \int g(x) = 2\tan x - 2x$

area under $f(x)$

$\left[3\tan x\right]_0^1 \quad (3\tan 1) - (3\tan 0) =$ area

area $= 4.67$ unit^2

area under $g(x)$

$\left[2\tan x - 2x\right]_0^1 \quad (2\tan 1 - 2) - (2\tan 0 - 0)$

area $= 1.11$ unit^2

area in grey = area $f(x)$ − area $g(x)$

area in grey $= 3.5574$ unit^2

$\boxed{\text{area} = 3.56 \text{ unit}^2}$

QUESTION TWO

(a) Find $\int(3x^4+4)^2\,\mathrm{d}x$.

$(3x^4+4)(3x^4+4)$

$= 9x^8 + 24x^4 + 16$

$= x^9 + 4.8x^5 + 16x$

(b) Find the value of k, given that $\int_k^{16} 3\sqrt{x}\,\mathrm{d}x = 112$.

You must use calculus and show the results of any integration needed to solve the problem.

(c) Consider the differential equation $\frac{dy}{dx} = 12y^2e^{3x}$.

$\cancel{12y^{-3}}$ $\frac{1}{12}y^{-2}$

$\cancel{-12y^{-1}}$

$\cancel{\frac{-12}{y}}$ $\frac{-1}{12y}$

Given that $y = 0.5$ when $x = 0$, find the value of y when $x = \frac{1}{3}$.

You must use calculus and show the results of any integration needed to solve the problem.

$$\int \frac{1}{12y^2} \times dy = \int e^{3x} \times dx$$

$$\frac{-1}{12y} = \frac{e^{3x}}{3} + c$$

$$\frac{-3}{12y} = e^{3x} + c \qquad -3 = e^{3x} \times 12y$$

$$12y = -\frac{3}{e^{3x}} \qquad y = -\frac{1}{4e^{3x}} + c$$

$$0.5 = \frac{-1}{4e^{3(0)}} + c$$

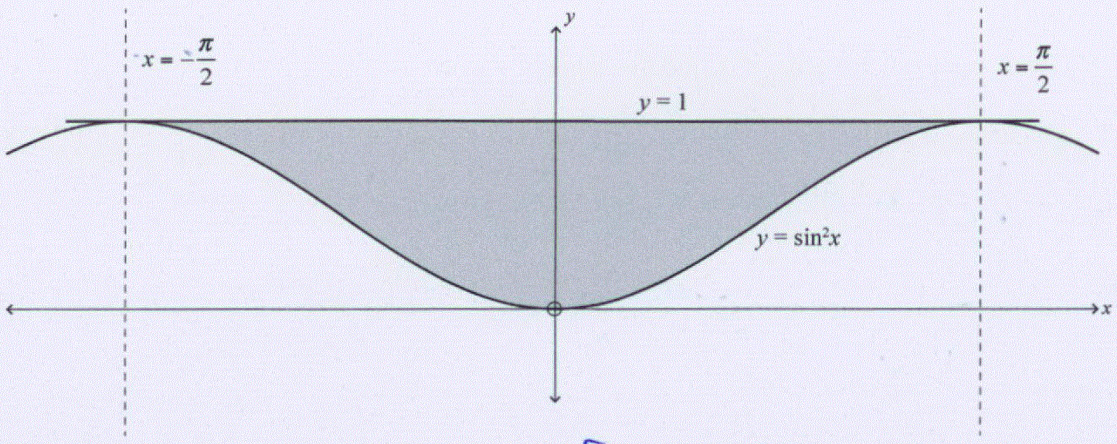
$$0.5 = -\frac{1}{4} + c \qquad c = .0.75$$

$$y = -\frac{1}{4e^{3x}} + 0.75$$

$$y = -\frac{1}{4e^{3(\frac{1}{3})}} + 0.75$$

$$\boxed{y = 0.658} \quad \text{or} \quad \boxed{\frac{1}{3}, 0.658}$$

(d) The graph below shows part of the graph of the function $y = \sin^2 x$.



Find the shaded area enclosed between the lines $\overset{f(x)}{y = \sin^2 x}$, $\overset{g(x)}{y = 1}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$.

You must use calculus and show the results of any integration needed to solve the problem.

$Q(x) = g(x) - f(x) \qquad Q(x) = 1 - \sin^2 x$

$Q(x) = 1 - \left(-\frac{1}{2}(\cos 2x - 1)\right)$

$Q(x) = 1 + \frac{1}{2}(\cos 2x - 1)$

$Q(x) = 1 + \frac{1}{2}\cos 2x - \frac{1}{2}$

$Q(x) = x + \frac{1}{4}\sin 2x - \frac{1}{2}x$

$Q(x) = \frac{1}{2}x + \frac{1}{4}\sin 2x$

$\left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$\left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \underbrace{\cancel{\frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)}}_{0}\right) - \left(\frac{1}{2}\left(-\frac{\pi}{2}\right) + \underbrace{\cancel{\frac{1}{4}\sin 2\left(-\frac{\pi}{2}\right)}}_{0}\right)$

$\boxed{\text{area} = \frac{1}{2}\pi \ \text{unit}^2}$

(e) The mass, M, of a spherical object, with radius p, can be approximated by

$$M = \int_0^p 4\pi r^2 \frac{a}{\left(1 + br^3\right)} \mathrm{d}r$$

where a, b, and p are all positive constants.

Using this formula, find an expression for the mass, M, of a spherical object, giving your answer in terms of a, b, p, and π.

$4\pi r^2 \int_0^p \frac{a}{1+br^3} dr$

QUESTION THREE

$3e^{-4x}$

(a) Find $\int\left(e^{2x}+\frac{3}{e^{4x}}\right)dx$.

$\frac{e^{2x}}{2} - \frac{3}{4e^{4x}} + c$

(b) Solve the differential equation $\frac{dy}{dx}=\frac{5}{4x-3}$, given that $y = 10$ when $x = 1$.

You must use calculus and show the results of any integration needed to solve the problem.

(c) Find the value of m, given that $\int_{-1}^{m}\left(\frac{4x+5}{2x+3}\right)dx = 2m$.

$$\frac{2(2x+3)-1}{(2x+3)}$$

$$2 - \frac{1}{2x+3}$$

$$\Big[2x - \ln|2x+3|\Big]_{-1}^{m} = 2m$$

$$\left(2m - \ln|2m+3|\right) - \left(-2 - \ln|1|\right) = 2m$$

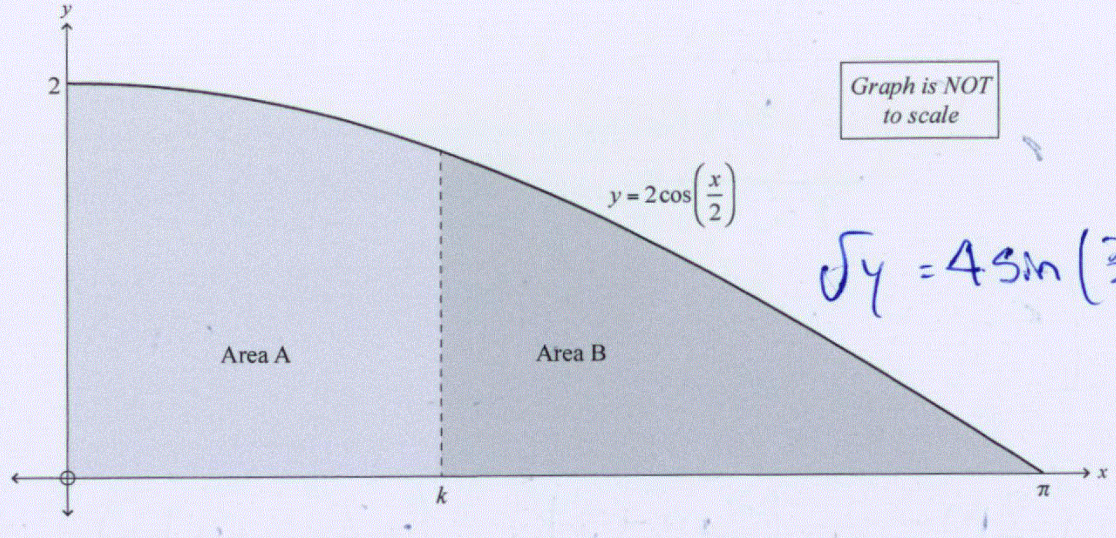
$$\cancel{2m} - \ln|2m+3| + 2 = \cancel{2m}$$

$$2 = \ln|2m+3| \qquad e^2 = 2m+3$$

$$\frac{e^2-3}{2} = m \qquad \boxed{m = 2.19}$$

Question Three continues on the next page.

(d) The graph below shows the function $y = 2\cos\left(\frac{x}{2}\right)$.



$\int y = 4\sin\left(\frac{x}{2}\right)$

Find the value of k so that the shaded Area A will be equal to the shaded Area B.

You must use calculus and show the results of any integration needed to solve the problem.

$$\left[4\sin\left(\frac{x}{2}\right)\right]_0^k = \left[4\sin\left(\frac{x}{2}\right)\right]_k^\pi$$

$$4\sin\left(\frac{k}{2}\right) = \left(4 - 4\sin\frac{k}{2}\right)$$

$$4\sin\left(\frac{k}{2}\right) = 4 - 4\sin\left(\frac{k}{2}\right)$$

$$8\sin\left(\frac{k}{2}\right) = 4$$

$$\sin\left(\frac{k}{2}\right) = 0.5$$

$$\frac{k}{2} = \sin^{-1}(0.5)$$

$$k = 2\sin^{-1}(0.5)$$

$$\boxed{k = \frac{1}{3}\pi}$$

(e) A teacher makes a cup of coffee at the start of interval. The teacher leaves the cup of coffee in the staff room where the temperature is 18 °C.

After 30 minutes, the temperature of the cup of coffee is 50 °C, but the teacher believes that the coffee is still too hot to drink.

When the teacher returns again, after a further one hour, the cup of coffee has cooled down to a temperature of 30 °C.

$t=90$

The rate at which the temperature of the cup of coffee changes at any instant is proportional to the difference between the temperature of the cup of coffee, N, and the temperature of the room.

Write a differential equation that models this situation, and then solve it to calculate the temperature of the cup of coffee when it was made.

You must use calculus and show the results of any integration needed to solve the problem.

$\frac{dT_N}{dt} \propto (T_N - T_r)$

$T_N - N = T$

$\frac{dT_N}{dt} = r(T_N - T_r)$

$T_r = 18$

$\int \frac{1}{T - T_r} \times dT = \int r \times dt$

$\ln|T - T_r| = rt + c$

$T - T_r = Ae^{rt}$

$T - 18 = Ae^{rt}$

$50 - 18 = Ae^{r \times 30}$ $\quad 32 = Ae^{30r}$

$30 - 18 = Ae^{90 \times r}$ $\quad 12 = Ae^{90r}$

$\frac{32}{A} = e^{30r}$ $\quad 30r = \ln\left(\frac{32}{A}\right)$

$r = \frac{\ln\left(\frac{32}{A}\right)}{30}$

$12 = Ae^{90 \times \frac{\ln\left(\frac{32}{A}\right)}{30}}$

$12 = Ae^{3\ln\left(\frac{32}{A}\right)}$

$\frac{12}{A} = e^{3\ln\left(\frac{32}{A}\right)}$

$\ln\left(\frac{12}{A}\right) = 3\ln\left(\frac{32}{A}\right)$

$A = 52.25578$ $\quad r = -0.016347$

$T - 18 = 52.3e^{-0.0163t}$

at $t = 0$

$T - 18 = 52.3e^{-0.0163(0)}$

$T = 70.3$

$\boxed{\text{initial temp} = 70.3°C}$

Excellence

Subject: Calculus

Standard: 91579

Total score: 21

Q	Grade score	Marker commentary
One	E8	Candidate correctly calculated the area between two curves by using a trigonometric identity and then integrating this expression. Candidate showed their working in a clear, correct logical manner.
Two	M5	Candidate calculated the area between a line and a curve by correctly using a trigonometric identity, and then integrating this expression correctly.
Three	E8	Candidate successfully formed and solved a differential equation from a written expression. They then used this expression to solve a problem using correct mathematical working.