2022 NCEA Assessment Report

Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Subject: **Calculus**

Level: **3**

Standards: **91577, 91578, 91579**

Part A: Commentary

The 2022 papers followed a similar format to recent years' papers. Overall candidate performance was also of a similar standard, which was pleasing to see considering the disruptions to the Candidates' education in the last couple of years. Many candidates displayed the ability to solve problems in both conventional and also in innovative methods. Unfortunately, there were also candidates who made fundamental mathematical errors of various natures, which showed lack of understanding and / or carelessness.

Common points made by panel leaders were:

- Candidates are encouraged to set out their working in a clear, logical, and systematic manner. This is a real advantage when solving the problems, providing the required evidence for the Examiner, as well as helping the candidate develop their strategy as they progress through the problem. This is beneficial particularly in problems which require an extended chain of reasoning.
- Candidates need to ensure that they read carefully the information provided in the question, which often provides helpful guidance and critical information, as well as helping the candidate recognise what the requirement of the question is. For example, some candidates provided only the *x*-coordinate when the question required the coordinates of a point, or provided only the gradient of normal when they were asked to give the equation of the normal.
- Candidates, whatever their target grade, are encouraged to attempt all parts of all questions. All question parts provide opportunities for success. It is not advisable for those candidates attempting to gain an overall Excellence grade to attempt only the questions that they believe to be 'excellence' questions. This strategy is not advisable, as an introduced error often leads to little reward. A large proportion of candidates who adopted this approach ended up with results that are well below their target grade. The Examiner carefully ensures that there is sufficient time available in the examination to attempt all question parts. Conversely, the weaker candidates often can gain valuable points in a question by completing part answers for the more challenging questions.
- Candidates who were not successful often attempted only one or two parts of each question. There are many opportunities to gain credit for correct working in all problems in all question parts, even though some candidates may struggle to solve the whole problem completely. Candidates are encouraged to answer at least a small portion of all question parts of the majority of paper in order to maximise their chance of success.
- Candidates need to understand that "Correct Answer Only" responses directly from the graphic calculator, do not provide evidence of relational thinking, and ensure that any incorrect working or part-solution should be crossed.

Part B: Report on standards

Standard number 91577 Apply the algebra of complex numbers.

Examination content and assessment specifications

This paper gave candidates multiple opportunities to display their understanding of the material outlined in the standard, namely

- quadratic and cubic equations with complex roots
- Argand diagrams
- polar and rectangular forms
- manipulation of surds
- manipulation of complex numbers
- loci
- de Moivre's theorem
- equations of the form $z^n = r\ \mathrm{cis}\Theta$, or $z^n = a + b\,\mathrm{i}$ where a and b are real, and n is a positive integer.

A study of previous years' examination papers and schedules will show that there are many skills which are regularly assessed. Problems related to the many and varied aspects of complex algebra need to be thoroughly understood and learnt for the higher levels of success in this achievement standard. In particular, questions requiring the application of de Moivre's theorem occur regularly, so candidates should be thoroughly prepared for this type of problem.

The requirement to show an answer in terms of a specific variable, or in in a specific form, is either not understood or is ignored by some candidates. If the question stipulates that the solution needs to be given in a particular format then this guidance needs to be followed, e.g. giving a complex number in the rectangular form or polar form.

Many candidates lacked the necessary knowledge and understanding in order to form the expression to solve the problem that resulted, or understand the algebra required to make progress toward an answer.

Candidates should realise that an explanation, conclusion, or interpretation at the end of a calculation is often a necessary and important part of the solution, and should be included in a solution

This standard does not cover as much content as the other two external papers. As a result, the paper is more predictable than the other two, and an organised revision programme which paid particular attention to previous papers would pay dividends.

Standard-specific observations

There are fundamental skills that candidates who want to achieve success in this standard must have the confidence in recognising and using. The ability to multiply and divide complex numbers in both rectangular and polar form is the most fundamental aspect of this standard. The need for multiplying by a conjugate fraction was required several times in this exam. It should be a skill that is thoroughly rehearsed by candidates in its various forms. The skill of understanding the modulus of a complex expression was required several times in the paper. Candidates need to be very familiar with the important definitions and related terminology that need to be recognised and utilised, e.g. magnitude, modulus, argument, conjugate, Argand diagram.

Application of de Moivre's theorem is another cornerstone technique. Candidates need to be more careful with their answers for questions requiring the use of de Moivre's theorem when finding solutions to an equation. Candidates need to ensure they calculate the correct argument when converting the original equation to polar form. If this initial angle is incorrect, then so is all the following work. This type of question needs to be given to setting up a general solution with a correct initial angle. Candidates had trouble representing the modulus correctly, so all other work done in the question was of no use. Using a sketch Argand diagram is recommended to illustrate in which quadrant the point is lying, without relying on a graphical calculator solution.

Grade related bullet points

Candidates who were awarded **Achievement** commonly:

- solved an equation by using completing the square method or the quadratic formula
- manipulated complex numbers in either Polar or Rectangular form
- showed an understanding of what the argument and modulus of a complex number were
- rationalised a denominator correctly
- simplified expressions involving surds
- represented a complex number on an Argand diagram
- identified real and imaginary terms in an expression and could group them correctly
- solved quadratics with unknowns as constants (either factorising or completing the square)
- understood the difference between and use the factor theorem and remainder theorem.

Candidates whose work was assessed as **Not Achieved** commonly:

- did not demonstrate basic algebra skills needed to solve, simplify, expand, and factorise
- displayed little understanding of the process for completing the square
- did not rationalise a denominator correctly
- did not find the modulus of a complex number
- did not manipulate complex numbers in either Polar or Rectangular form
- converted a complex number from one form to another (rectangular to Polar or vice versa) before performing calculations with complex numbers

- did not demonstrate how to convert to Polar form, and / or understand what an argument is
- did not demonstrate problem solving techniques; instead relied too heavily on the usage of a graphics calculator
- did not understand how to interpret de Moivre's Theorem
- did not understand how to interpret complex numbers represented on an Argand diagram
- did not understand how to use the Remainder Theorem.

Candidates who were awarded **Achievement with Merit** commonly:

- distinguished between the factors and solutions of an equation
- understood the meaning of modulus and argument, and were able to express statements using these features correctly
- expanded quadratics involving surds or imaginary numbers
- understood how to use de Moivre's Theorem and could apply it correctly
- understood the meaning of "purely real" or "purely imaginary" complex numbers and could form and solve the equations that resulted
- substituted $z = x + iy$ into equations, and manipulated them to separate them into real and imaginary parts
- solved problems involving the discriminant in order to find unknown constants
- recognised the definitions and terminology of complex numbers
- used appropriate algebraic methods to find complex roots of a cubic polynomial, given one imaginary root
- identified that a complex number with equal positive real and imaginary parts has an argument of $\pi/4$.

Candidates who were awarded **Achievement with Excellence** commonly:

- used their algebra skills to accurately solve an equation, involving quotients of complex numbers, without unnecessary or confusing statements in their working
- demonstrated problem-solving skills required to group real and imaginary terms, and could apply the correct algebra to them
- understood what the modulus symbol required, and how it can be applied in a formal proof
- explored problems that not all solutions will be valid and solving accordingly
- completed the required proof by making connections between real and imaginary parts and completing the square
- communicated their thinking clearly and accurately about what they were doing while completing multi-step problems
- provided clear, logical, and easy-to-follow working out, reducing the chance of numerical or algebraic errors.

Standard number 91578 Apply differentiation methods in solving problems.

Examination content and assessment specifications

This paper gave candidates multiple opportunities to display their understanding of the material outlined in the standard namely:

- derivatives of power, exponential, and logarithmic (base e only) functions
- derivatives of trigonometric (including the reciprocal trigonometric) functions
- optimisation
- equation of a normal
- maxima and minima and points of inflection
- related rates of change
- derivatives of parametric functions
- chain, product, and quotient rules
- properties of graphs (limits, differentiability, continuity, concavity).

Candidates must ensure that they are fully confident in their own ability to be able to differentiate all various forms of the functions, including when the use of the chain, product and quotient rules are necessary. Many of the candidates displayed the necessary understanding to be able to interpret the application using differentiation methods but were let down by errors in their methods of differentiation. Success can only be awarded if the careless mistakes are avoided.

In order to solve the application of differentiation problems candidates require sound algebra skills, which come from continual practice from a wide variety of possibilities. Many candidates are not sufficiently strong enough to solve a simple quadratic equation or equations involving the use of exponentials and logarithms. Candidates should not be relying solely on their graphical calculators to solve such equations as the inclusion of unknown constants will generally block this route. For all levels of success, both necessary skills of differentiation and algebra are necessary.

Modelling for optimisation problems continues to be a skill that only the best candidates can handle. Candidates need to work with their supporters in order to find ways to gain greater confidence in these types of problems. The optimisation problem, involving speed / distance / time, (Question Three, part e) was too hard for all but the very top candidates. Candidates need a lot more practice at understanding how to deal with creating appropriate models within an optimisation problem.

Many candidates were confused when the problem connected their differentiation and co-ordinate geometry knowledge. (Question One, part c) These types of problems should be included in all teaching programmes. It was evident from the errors seen that candidates were not sufficiently confident and experienced in this type of problem-solving.

Standard-specific observations

Good algebraic skills are crucial for success in this standard: expanding, factorising, simplifying expressions, manipulating algebraic fractions, and solving various equations, especially quadratic equations but also equations involving exponential, logarithmic and trigonometric functions. Similarly, candidates should be aware that the square root of a value will always lead to two possible solutions, however, one of which may not be valid.

The correct use of brackets and accurate mathematical statements remains a challenge for many Calculus candidates. The ability to avoid making careless errors or having an effective self-checking system to eliminate any careless mistakes is important.

Many candidates were unable to differentiate correctly to find the first and second derivatives of the function in Question One part e as errors were confused by the constant and then failed to recognise that the subsequent differentiations required the use of the product rule.

Candidates need to be aware and knowledgeable regarding completing a proof in a mathematical manner. Many candidates were able to differentiate successfully but did not produce their solution in a sufficiently formal manner.

Grade related bullet points

Candidates who were awarded **Achievement** commonly:

- used the chain rule, product rule and quotient rule correctly in combination with power functions, trigonometric functions, parametric functions, exponential and logarithmic functions
- solved quadratic equations resulting from differentiating
- used differentiation to find the gradient of tangents
- solved the velocity function from the given displacement function
- used an appropriate model to solve an optimisation problem and then correctly differentiated their model
- solved the two components of a related rates of change problem and then correctly differentiated
- calculated the *x*-coordinate correctly for any stationary points
- recognised features of gradient, differentiability and limit from a graph
- used appropriate interval notation to describe when a piecewise function had zero gradient
- understood that the limit of a function could exist where there was a “hole” in the graph.

Candidates whose work was assessed as **Not Achieved** commonly:

- did not apply the chain rule, product rule or quotient rule in combination with power, trigonometric functions, parametric functions, exponential, and logarithmic functions correctly
- did not recognise when it was necessary to apply the product and quotient rules for differentiation
- did not differentiate product and quotient formulae correctly
- did not solve a resulting equation algebraically, even though they had found the correct derivative
- did not correctly factorise and solve quadratic expressions, even though they had found the correct derivative

- demonstrated poor algebraic skills such as in accurate expanding, factorising, cancelling of factors, manipulating algebraic fractions, careless errors, providing only one solution for the square root of a value
- did not recognise features of gradient, differentiability, and limits from a graph
- did not actually answer the requirements of the question.

Candidates who were awarded **Achievement with Merit** commonly:

- solved equations involving logarithmic and exponential functions correctly to find the coordinates of stationary points, demonstrating confident algebraic skills
- used the product rule, quotient rule and chain rule correctly
- provided the required equation of a normal and used it to find the x-axis intercept
- solved a simple related rates problem
- demonstrated knowledge of kinematics in solving a problem
- solved the co-ordinates of a point on a parametric function graph where the gradient has a specified value
- demonstrated reliable and accurate algebra skills when solving problems
- demonstrated the procedure to set up models for optimisation questions, in terms of one variable, before correctly differentiating it and then solving the resulting equations
- solved second differentials of an exponential function
- manipulated surds correctly, resulting from differentiating
- located and identified the nature of stationary points, with a variety of options of methods .

Candidates who were awarded **Achievement with Excellence** commonly:

- demonstrated high quality algebraic skills when solving problems resulting from differentiating
- provided appropriate models and then interpreted them correctly to solve the problem using differentiation
- demonstrated high-quality understanding of kinematics
- used the second derivative of functions when solving problems
- linked differentiation and co-ordinate geometry methods when proving a result, utilising precise and accurate algebraic methods
- used the quotient rule to find the derivative of a rational function and solve the resulting quadratic equation to find the values of *x* for which the function had stationary values, identifying their nature
- completed a proof using logical, clear working and communicating each step of a 'proof' question with sufficient detail
- formed and interpreted an optimisation problem to find the best position for Megan to cycle across a park in order to minimise the time for her journey.

Standard number 91579 Apply integration methods in solving problems.

Examination content and assessment specifications

This paper gave candidates multiple opportunities to display their understanding of the material outlined in the standard namely:

- integrating power, polynomial, exponential (base e only), trigonometric, and rational functions
- reverse chain rule, trigonometric formulae
- rates of change problems
- areas under or between graphs of functions, by integration
- finding areas using numerical methods, e.g. the rectangle or trapezium rule
- differential equations of the forms $y' = \mathrm{f}(x)$ or $y'' = \mathrm{f}(x)$ for the above functions or situations where the variables are separable (e.g. $y' = ky$) in applications such as growth and decay, inflation, Newton's Law of Cooling and similar situations.

While the ability to integrate the types of functions listed in the bullet points above is critical to success, so is having the algebra skills required to solve the related problems. These algebra skills are necessary for all three external standards in the Calculus course and their constant practice and reinforcement should be a focus for all candidates.

Standard-specific observations

Candidates should be strongly discouraged from only attempting the "Excellence" part of each question. It was very common for candidates who did this to make an error on one or more of these.

Candidates should be encouraged to use the formal and correct terminology throughout the solution to a problem in order to avoid careless errors and to guide them through each step of the necessary working and calculations.

Candidates are reminded that evidence of the correct use of integration methods must be clearly demonstrated; answers from a graphical calculator are unlikely to attain higher than an “achieved” grade, at best.

Within the teaching and learning programme it needs to be stressed that candidates must correctly use the constant of integration. The constant of integration should not be omitted and, additionally, its value need not necessarily be zero.

Integrating $\sin^2 2x$ to give $\frac{1}{3}\sin^3 2x$ was unfortunately quite common. Also, candidates frequently omitted a factor of the integration. Both of these errors led to an incorrect integration so no credit could be given for this question even for candidates who managed to arrive at the correct solution.

Grade related bullet points

Candidates who were awarded **Achievement** commonly:

- integrated basic functions correctly, including polynomial functions, rational functions, exponential functions, trigonometric functions
- manipulated functions into a form which could be integrated
- attempted the majority of questions, in particular integrating any functions in the paper that needed to be integrated
- used the Trapezium rule correctly
- understood the fundamental theorem of integration
- provided the constant of integration, given the necessary information.

Candidates whose work was assessed as **Not Achieved** commonly:

- did not demonstrate sufficient understanding to recognise the appropriate method and consequent accurate integration when integrating power, polynomial, exponential, trigonometric, and rational functions
- did not algebraically rearrange or manipulate a function so that it was in a suitable form which could then be integrated
- did not show appropriate usage of the Trapezium Rule formula in order to complete a numerical method
- did not recognise how integration could be used to evaluate an area
- were not sufficiently confident with the rules of indices so that errors appeared when manipulating $\sqrt{x}$ into an alternative appropriate format
- did not identify the correct use differentiation and integration methods in problem solving.

Candidates who were awarded **Achievement with Merit** commonly:

- integrated a rational function by using either long division or substitution
- used trigonometric identities correctly in order to be able to apply an appropriate integration method
- solved differential equations by separating variables, including calculating the arbitrary constant
- recognised when to use the natural logarithm to integrate
- evaluated an area that lies between two functions
- applied integration methods in relation to kinematics
- integrated $\sin^2 2x$

Candidates who were awarded **Achievement with Excellence** commonly:

- carried out appropriate algebraic manipulations in order to attain an expression suitable to integrate
- demonstrated the knowledge and understanding in order to solve problems involving areas
- applied understanding in order to work with exact values, manipulating logarithms
- demonstrated understanding to use correct and appropriate mathematical statements

- demonstrated the ability to form a differential equation for a contextual problem, correctly separated the variables and then used the appropriate information to find the value of the constants and hence solve the problem.